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Parallelepiped Inequality into 2-Normed Space and its Consequences

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Abstract: *The concept of 2-norm is two-dimensional analogy of the concept of norm and is given by S. Gähler in* 1965*. So, the 2-norm axioms consist an axiom which is analogy of triangle inequality into normed space, and is called as parallelepiped inequality. In this paper are proved a few inequalities, which in fact are consequences of parallelepiped inequality and are analogy of appropriate inequalities into normed space.*

Keywords: *2-norm, 2-inner product, parallelepiped inequality.2010*

Mathematics Subject Classification. 46B20, 46B99, 51M16

1. Introduction

Let L be a real vector space with dimension greater than 1 and, $\|\cdot,\cdot\|$ be a real function defined on $L\times L$ which satisfies the following:

a) $\|x,y\|\geq 0$, for all $x,y\in L$ and $\|x,y\|=0$ if and only if the set $\{x,y\}$ is linearly dependent;

b) $\|x,y\|=\|y,x\|$, for all $x,y\in L$;

c) $\|\alpha x,y\|=|\alpha|\cdot\|x,y\|$, for all $x,y\in L$ and for each $\alpha\in\mathbf{R}$,

d) $\|x+y,z\|\leq\|x,z\|+\|y,z\|$, for all $x,y,z\in L$.

Function $\|\cdot,\cdot\|$ is called as 2-*norm of L*, and $(L,\|\cdot,\cdot\|)$ is called as vector 2-normed space ([7]).

Let $n>1$ be a positive integer, L be a real vector space, $\dim L\geq n$ and $(\cdot,\cdot|\cdot)$ be a real function on $L\times L\times L$ such that:

i) $(x,x|y)\geq 0$, for all $x,y\in L$ and $(x,x|y)=0$ if and only if a and b are linearly dependent;
ii) $(x,y|z)=(y,x|z)$, for all $x,y,z\in L.$,
iii) $(x,x|y)=(y,y|x)$, for all $x,y\in L$;
iv) $(\alpha x,y|z)=\alpha(x,y|z)$, for all $x,y,z\in L.$ and for each $\alpha\in\mathbf{R}$; and
v) $(x+x_1,y|z)=(x,y|z)+(x_1,y|z)$, for all $x_1,x,y,z\in L$.

Function $(\cdot,\cdot|\cdot)$ is called as 2-*inner product*, and $(L,(\cdot,\cdot|\cdot))$ is called as 2-*pre-Hilbert space* ([6]).

Example 1. Let $(L,(\cdot,\cdot))$ be a real pre-Hilbert space. Then,

$$\|x,z\|=\begin{vmatrix}(x,x) & (x,z)\\ (x,z) & (z,z)\end{vmatrix}^{1/2},$$

for each $x,z\in L$, defines a so called standard 2-norm. Further, if $L=\mathbf{R}^3$ with ordinary inner product and the vectors $x,y,z\in\mathbf{R}^3$ are not pairwise linearly dependent then $\|x,z\|$, $\|y,z\|$,

$\| x+y,z \|$ are equal to the areas of the parallelograms constructed with the vectors x and z,y and $z,x+y$ and z, respectively. Inequality

$$\| x+y,z \| \leq \| x,z \| + \| y,z \|, \tag{1}$$

is equivalent to the

$$0 \leq 1-\cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma\,, \tag{2}$$

for $\measuredangle(x,y)=\alpha$, $\measuredangle(y,z)=\beta$ and $\measuredangle(z,x)=\gamma$. Geometrical interpretation of the inequality (1) states following: the sum of the areas of two adjacent faces is greater or equal to the area of diagonal intersection which is placed between these two faces. So, the inequality (1) is the analogy of the triangle inequality into normed space and is called as *parallelepiped inequality.*

2. Parallelepiped Inequality

Lemma 1. Let $(L,\| \cdot,\cdot \|)$ be a 2-normed space and $z,x_i \in L$, $i=1,2,...,n$.

a) If $\alpha_i \geq 0, i=1,2,...,n$, then

$$\| \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n, z \| \leq \alpha_1 \| x_1,z \| + \alpha_2 \| x_2,z \| + ... + \alpha_n \| x_n,z \|. \tag{3}$$

б) If $\alpha_1 > 0$ and $\alpha_i \leq 0$, for $i=2,3,...,n$, then

$$\| \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n, z \| \geq \alpha_1 \| x_1,z \| + \alpha_2 \| x_2,z \| + ... + \alpha_n \| x_n,z \|. \tag{3'}$$

Proof. a) Directly follows by the inequality (1), axiom c) of 2-norm and the principle of mathematical induction.

б) The inequality (3) implies the following

$$\begin{aligned}\| \alpha_1 x_1, z \| &= \| \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n - (\alpha_2 x_2 + ... + \alpha_n x_n), z \| \\ &\leq \| \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n, z \| + \| -(\alpha_2 x_2 + ... + \alpha_n x_n), z \| \\ &\leq \| \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n, z \| + (-\alpha_2) \| x_2,z \| + ... + (-\alpha_n) \| x_n,z \|\end{aligned}$$

which is equivalent to the inequality (3').

Remark 1. Substituting $\alpha_i = 1, i=1,2,...,n$ into the inequality (3), we get that for each $z,x_i \in L$, $i=1,2,...,n$ the following inequality is satisfied

$$\| x_1 + x_2 + ... + x_n, z \| \leq \| x_1,z \| + \| x_2,z \| + ... + \| x_n,z \|. \tag{4}$$

Lemma 2. Let $(L,\| \cdot,\cdot \|)$ be a 2-normed space and $z,x_i \in L$, $i=1,2,...,n$. Then

$$\| x_1 + x_2 + ... + x_n, z \| = \| x_1,z \| + \| x_2,z \| + ... + \| x_n,z \|, \tag{5}$$

if and only if

$$\| \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n, z \| = \alpha_1 \| x_1,z \| + \alpha_2 \| x_2,z \| + ... + \alpha_n \| x_n,z \|, \tag{6}$$

for each $\alpha_i > 0, i=1,2,...,n$.

Proof. If the equality (6) is satisfied for each $\alpha_i > 0, i=1,2,...,n$, then letting $\alpha_i = 1, i=1,2,...,n$ we get the equality (5).

Conversely, let (5) be satisfied and let $\alpha_i > 0, i=1,2,...,n$. Without any restriction of generality we may take $\alpha_1 = \max\limits_{1\leq i\leq n} \alpha_i$. Then the equality (5) and lemma 1 imply the following

$$\begin{aligned}\sum_{i=1}^{n}\alpha_i \|x_i,z\| &= \alpha_1\sum_{i=1}^{n}\|x_i,z\| - \sum_{i=1}^{n}(\alpha_1-\alpha_i)\|x_i,z\| \\ &= \alpha_1\|\sum_{i=1}^{n}x_i,z\| - \sum_{i=1}^{n}(\alpha_1-\alpha_i)\|x_i,z\| \\ &\le \|\alpha_1\sum_{i=1}^{n}x_i,z\| - \|\sum_{i=1}^{n}(\alpha_1-\alpha_i)x_i,z\| \\ &\le \|\alpha_1\sum_{i=1}^{n}x_i - \sum_{i=1}^{n}(\alpha_1-\alpha_i)x_i,z\| \\ &= \|\sum_{i=1}^{n}\alpha_i x_i,z\|.\end{aligned}$$

Finally, the last inequality and the inequality (3) imply the inequality (6).

Let $x,y\in L$ be non-zero elements and $V(x,y)$ denotes a subspace of L generated by the vectors x and y. The 2-normed space $(L,\|\cdot,\cdot\|)$ is called as strictly convex if $\|x,z\|=\|y,z\|=\|\frac{x+y}{2},z\|=1$ and $z\notin V(x,y)$, for $x,y,z\in L$, imply $x=y$ ([3]). The condition for which the equivalent equalities (5) and (6) are satisfied in strictly convex space is given by lemma 3. The following theorem holds true:

Theorem 2([4]). The 2-normed space $(L,\|\cdot,\cdot\|)$ is strictly convex if and only if $\|x+y,z\|=\|x,z\|+\|y,z\|$ and $z\notin V(x,y)$, for $x,y,z\in L$ imply $y=\alpha x$ for some $\alpha>0$.

Lemma 3. Let $(L,\|\cdot,\cdot\|)$ be a strictly convex 2-normed space and the vectors $z,x_i\in L$, $i=1,2,...,n$ are such that the sets $\{x_i,z\}$, $i=1,2,...,n$ are linearly independent. Then the equalities (5) and (6) are equivalent with the equalities

$$\frac{x_1}{\|x_1,z\|}=\frac{x_2}{\|x_2,z\|}=...=\frac{x_n}{\|x_n,z\|}. \tag{7}$$

Proof. If the equalities (7) are satisfied, then for each $\alpha_i>0$, $i=1,2,...,n$ holds true the following

$$\begin{aligned}\|\sum_{i=1}^{n}\alpha_i x_i,z\| &= \|\sum_{i=1}^{n}\alpha_i\|x_i,z\|\frac{x_i}{\|x_i,z\|},z\| = \|\sum_{i=1}^{n}\alpha_i\|x_i,z\|\frac{x_1}{\|x_1,z\|},z\| \\ &= \|(\sum_{i=1}^{n}\alpha_i\frac{\|x_i,z\|}{\|x_1,z\|})x_1,z\| = (\sum_{i=1}^{n}\alpha_i\frac{\|x_i,z\|}{\|x_1,z\|})\|x_1,z\| = \sum_{i=1}^{n}\alpha_i\|x_i,z\|,\end{aligned}$$

i.e. the equality (6) holds true.

Conversely, let (5) hold. For each $i=2,3,...,n$ is true that $\|x_1+x_i,z\|\le\|x_1,z\|+\|x_i,z\|$. Furthermore,

$$\begin{aligned}\|x_1+x_i,z\| &\ge \|\sum_{k=1}^{n}x_k,z\| - \|\sum_{k\ne 1,i}x_k,z\| = \sum_{k=1}^{n}\|x_k,z\| - \|\sum_{k\ne 1,i}x_k,z\| \\ &\ge \sum_{k=1}^{n}\|x_k,z\| - \sum_{k\ne 1,i}\|x_k,z\| = \|x_1,z\|+\|x_i,z\|,\end{aligned}$$

so $\|x_1+x_i,z\|=\|x_1,z\|+\|x_i,z\|$. But, L is strictly convex, and using facts stated in Theorem 2 follows $x_1=\alpha_i x_i$, for $i=2,3,..,n$. So, $\|x_1,z\|=\alpha_i\|x_i,z\|$, for $i=2,3,..,n$, i.e. $\alpha_i=\frac{\|x_1,z\|}{\|x_i,z\|}$, for $i=2,3,..,n$. Hence, $\frac{x_1}{\|x_1,z\|}=\frac{x_i}{\|x_i,z\|}$, for $i=2,3,..,n$, i.e. the equalities (7) are satisfied.

Theorem 3. Let $(L,\|\cdot,\cdot\|)$ be a 2-normed space. For each $x,y,z\in L$ the followings are satisfied

$$|\|x,z\|-\|y,z\||\leq\|x+y,z\|+\|x-y,z\|-\|x,z\|-\|y,z\|\leq\min\{\|x+y,z\|,\|x-y,z\|\} \quad (8)$$

and

$$|\|x,z\|-\|y,z\||\leq\|x,z\|+\|y,z\|-|\|x+y,z\|-\|x-y,z\||. \quad (9)$$

Proof. The parallelepiped inequality implies

$$\|x+y,z\|+\|x-y,z\|-\|x,z\|-\|y,z\|\leq\|x-y,z\|$$
$$\|x+y,z\|+\|x-y,z\|-\|x,z\|-\|-y,z\|\leq\|x+y,z\|$$

So,

$$\|x+y,z\|+\|x-y,z\|-\|x,z\|-\|y,z\|\leq\min\{\|x-y,z\|,\|x+y,z\|\},$$

i.e. the right side inequality of (8) is satisfied.

Further, again using the parallelepiped inequality we get

$$2\|x,z\|=\|x+y+(x-y),z\|\leq\|x+y,z\|+\|x-y,z\|.$$
$$2\|y,z\|=\|x+y-(x-y),z\|\leq\|x+y,z\|+\|x-y,z\|,$$

So,

$$2\max\{\|x,z\|,\|y,z\|\}\leq\|x+y,z\|+\|x-y,z\|. \quad (10)$$

On the other hand

$$\|x,z\|+\|y,z\|+|\|x,z\|-\|y,z\||=2\max\{\|x,z\|,\|y,z\|\}. \quad (11)$$

Finally, the equality (11) and inequality (10) imply the left side inequality of (8).

Again, from the parallelepiped inequality we get

$$\|x+y,z\|\leq\|x+y-(x-y),z\|+\|x-y,z\|=2\|y,z\|+\|x-y,z\|,$$

$$\|y-x,z\|\leq\|y-x-(x+y),z\|+\|x+y,z\|=2\|x,z\|+\|x+y,z\|.$$

So,

$$|\|x+y,z\|-\|x-y,z\||\leq 2\min\{\|x,z\|,\|y,z\|\}. \quad (12)$$

On the other hand

$$\|x,z\|+\|y,z\|-|\|x,z\|-\|y,z\||=2\min\{\|x,z\|,\|y,z\|\}. \quad (13)$$

Finally, the equality (13) and the inequality(12) imply the inequality (9).

3. The Second Type of Parallelepiped Inequality

Firstly, we will discuss the convex functions defined on a vector space L. Let L be a vector space and $C\subseteq X$ be a convex subset of X. The function $f:C\to\mathbf{R}$ is convex if for all $x,y\in C$ and for each $\alpha\in[0,1]$ the inequality $f(\alpha x+(1-\alpha)y)\leq\alpha f(x)+(1-\alpha)f(y)$ is satisfied. Furthermore, the function $f:C\to\mathbf{R}$ is convex if and only if for each $x_i\in C$, $i=1,2,...,n$ and for each $\alpha_i\geq 0$, $i=1,2,...,n$, such that $\sum_{i=1}^{n}\alpha_i=1$, the Jensen inequality holds true, i.e.

$$f(\sum_{i=1}^{n}\alpha_i x_i)\leq\sum_{i=1}^{n}\alpha_i f(x_i). \quad (14)$$

Lemma 4. If $(L,\|\cdot,\cdot\|)$ be a 2-normed space, then for each $z \in L$ and for each $p \geq 1$ the function $f_{p,z} : L \to \mathbf{R}$ defined as $f_{p,z}(x) = \| x, z \|^p$, $x \in L$ is convex.

Proof. Let $z \in L$ and $p \geq 1$. The function $f(t) = t^p$, $p \geq 1$ is convex on $[0,\infty)$. So, for each $t_i \geq 0, i = 1,2,...,n$ and for each $\alpha_i \geq 0$, $i = 1,2,...,n$, $\sum_{i=1}^{n} \alpha_i = 1$ is true following inequality

$$(\sum_{i=1}^{n} \alpha_i t_i)^p \leq \sum_{i=1}^{n} \alpha_i t_i^p \ . \tag{15}$$

Further, the parallelepiped inequality and the inequality (15) for $t_i = \| x_i, z \|$, imply

$$\| \sum_{i=1}^{n} \alpha_i x_i, z \|^p \leq (\sum_{i=1}^{n} \alpha_i \| x_i, z \|)^p \leq \sum_{i=1}^{n} \alpha_i \| x_i, z \|^p$$

for each $x_i \in L$, $i = 1,2,...,n$ and for each $\alpha_i \geq 0$, $i = 1,2,...,n$ such that $\sum_{i=1}^{n} \alpha_i = 1$, i.e. for the function $f_{p,z}$, Jensen inequality is satisfied. That means, the stated function is convex.

Theorem 4. Let $(L,\|\cdot,\cdot\|)$ be a 2-normed space. For all $z, x_1,..., x_n \in L$, $\alpha_i > 0, i = 1,2,...,n$ and $p \geq 1$ the following inequality holds true

$$\frac{\|x_1+x_2+...+x_n,z\|^p}{(\alpha_1+\alpha_2+...+\alpha_n)^{p-1}} \leq \frac{\|x_1,z\|^p}{\alpha_1^{p-1}} + \frac{\|x_2,z\|^p}{\alpha_2^{p-1}} + ... + \frac{\|x_n,z\|^p}{\alpha_n^{p-1}} \ . \tag{16}$$

Proof. Let $\alpha_i > 0, i = 1,2,...,n$ and $p \geq 1$. Then

$$r_i = \alpha_i (\sum_{i=1}^{n} \alpha_i)^{-1} > 0 \ \text{и} \ \sum_{i=1}^{n} r_i = 1 .$$

According to Lemma 4, for all $z, y_i \in L, i = 1,2,...,n$ is true that

$$\| r_1 y_1 + r_2 y_2 + ... + r_n y_n, z \|^p \leq r_1 \| y_1, z \|^p + r_2 \| y_2, z \|^p + ... + r_n \| y_n, z \|^p$$

i.e. holds true

$$\frac{\|\alpha_1 y_1+\alpha_2 y_2+...+\alpha_n y_n,z\|^p}{(\alpha_1+\alpha_2+...+\alpha_n)^{p-1}} \leq \alpha_1 \| y_1, z \|^p + \alpha_2 \| y_2, z \|^p + ... + \alpha_n \| y_n, z \|^p \ . \tag{17}$$

Finally, letting $y_i = \frac{x_i}{\alpha_i}, i = 1,2,...,n$ in the equality (17), we get the inequality (16).

Remark 2. Letting $n = 2$ in the inequality (16), we get the following

$$\frac{\|x_1+x_2,z\|^p}{(\alpha_1+\alpha_2)^{p-1}} \leq \frac{\|x_1,z\|^p}{\alpha_1^{p-1}} + \frac{\|x_2,z\|^p}{\alpha_2^{p-1}} , \tag{18}$$

Moreover, if $\alpha_1 = \alpha_2 = 1$ then follows

$$\| x_1 + x_2, z \|^p \leq 2^{p-1} (\| x_1, z \|^p + \| x_2, z \|^p) \ .$$

Now, letting $p = 2$ into the last inequality, we get other form of the parallelepiped inequality which is called as the second type of parallelepiped inequality.

$$\| x_1 + x_2, z \|^2 \leq 2(\| x_1, z \|^2 + \| x_2, z \|^2) \ . \tag{19}$$

Moreover, letting $p=2$, $x_1=ay_1$, $x_2=by_2$, $\alpha_1=\alpha a^2$ and $\alpha_2=\beta b^2$ in (18) we get the following inequality, which in fact is more general second type parallelepiped inequality

$$\frac{\|ay_1+by_2,z\|^2}{\alpha a^2+\beta b^2}\le\frac{\|y_1,z\|^2}{\alpha}+\frac{\|y_2,z\|^2}{\beta}. \tag{20}$$

Letting $a=b=\alpha=\beta=1$ we get the inequality (19).

Remark 3. Let $(L,(\cdot,\cdot\,|\,\cdot))$ be a 2-pre-Hilbert space. Theorem 1 [2] proves

$$\frac{\|ax+by,z\|^2}{\gamma}+\frac{\|\beta bx-\alpha ay,z\|^2}{\gamma\alpha\beta}=\frac{\|x,z\|^2}{\alpha}+\frac{\|y,z\|^2}{\beta}, \tag{21}$$

for all $x,y,z\in L$ and for all $a,b\in\mathbf{R}$, $\alpha,\beta>0$, $\gamma=\alpha a^2+\beta b^2$. But for all $x,y,z\in L$ and for all $a,b\in\mathbf{R}$, $\alpha,\beta>0$, $\gamma=\alpha a^2+\beta b^2$ holds $\frac{\|\beta bx-\alpha ay,z\|^2}{\gamma\alpha\beta}\ge 0$. So, in the 2-pre-Hilbert space, the equality (21) implies the inequality (20).

Let $x,y,z\in L$, $p\ge 1$ and consider the function $g:[0,1]\to\mathbf{R}$ defined by

$$g(t)=\|tx+(1-t)y,z\|^p,\ t\in[0,1].$$

The continuously of 2-norm implies the continuously of this function on $[0,1]$. The last in fact means, the function is integrable on $[0,1]$. But, Lemma 4 imply

$$g(t)=\|tx+(1-t)y,z\|^p\le t\|x,z\|^p+(1-t)\|y,z\|^p, \text{ for each } t\in[0,1].$$

So,

$$\int_0^1\|tx+(1-t)y,z\|^p\,dt\le\int_0^1(t\|x,z\|^p+(1-t)\|y,z\|^p)dt=\frac{\|x,z\|^p+\|y,z\|^p}{2}. \tag{22}$$

Let $t_1,t_2\in[0,1]$ and $\lambda\in[0,1]$. Using Lemma 4 once again, we get

$$\begin{aligned}g(\lambda t_1+(1-\lambda)t_2)&=\|(\lambda t_1+(1-\lambda)t_2)x+(1-\lambda t_1-(1-\lambda)t_2)y,z\|^p\\&=\|(\lambda t_1+(1-\lambda)t_2)x+(\lambda(1-t_1)+(1-\lambda)(1-t_2))y,z\|^p\\&=\|\lambda(t_1x+(1-t_1)y)+(1-\lambda)(t_2x+(1-t_2)y),z\|^p\\&\le\lambda\|t_1x+(1-t_1)y,z\|^p+(1-\lambda)\|t_2x+(1-t_2)y,z\|^p\\&=\lambda g(t_1)+(1-\lambda)g(t_2).\end{aligned}$$

This means that the function $g(t)$ is convex on $[0,1]$. Now, by Jensen integral inequality ([13], p.p. 13, Theorem 1) we get

$$\begin{aligned}\|\tfrac{x+y}{2},z\|^p&=\|\int_0^1(tx+(1-t)y)dt,z\|^p=g(\int_0^1(tx+(1-t)y)dt)\\&\le\int_0^1 g(tx+(1-t)y)dt=\int_0^1\|tx+(1-t)y,z\|^p\,dt.\end{aligned} \tag{23}$$

The inequalities (22) and (23) in fact are generalization of the J. E. Pečarić and S. S. Dragomir ([13], p.p. 485 (5.10)) inequalities at 2-normed space and $p\ge 1$. Finally, the above stated, i.e. the inequalities (22) and (23), implies the following Lemma.

Lemma 5. Let $(L,\|\cdot,\cdot\|)$ be a 2-normed space. For all $x,y,z\in L$ and for each $p\ge 1$ holds true

$$\|\tfrac{x+y}{2},z\|^p \le \int_0^1 \|tx+(1-t)y,z\|^p\,dt \le \tfrac{\|x,z\|^p+\|y,z\|^p}{2}. \tag{24}$$

Let X be a vector space, C be a convex subset of X, P_n be a set of all non-negative n– tuples $(p_1, p_2,..., p_n)$ so that $\sum_{i=1}^{n} p_i = 1$, $f: C \to \mathbf{R}$ is a convex function, $\mathbf{x} = (x_1, x_2,..., x_n) \in C$, $\mathbf{p} \in P_n$ and

$$J_n(f,\mathbf{x},\mathbf{p}) = \sum_{i=1}^{n} p_i f(x_i) - f(\sum_{i=1}^{n} p_i x_i) \ge 0, \tag{25}$$

be a normalized Jensen functional. In paper [5], for functional (25), S.S. Dragomir gave an elementary proof of the next Theorem about the bounds of normalized Jensen functional.

Theorem 5 [5]. If $\mathbf{p},\mathbf{q} \in P_n$, $q_i > 0$, for each $i = 1,2,...,n$ then

$$J_n(f,\mathbf{x},\mathbf{q}) \max_{1\le i\le n}\{\tfrac{p_i}{q_i}\} \ge J_n(f,\mathbf{x},\mathbf{p}) \ge J_n(f,\mathbf{x},\mathbf{q}) \min_{1\le i\le n}\{\tfrac{p_i}{q_i}\}. \tag{26}$$

Using this S. S. Dragomir's result, we will prove few inequalities into 2-normed space, which are analogy of the corresponding inequalities at normed space.

Theorem 6. Let $(L,\|\cdot,\cdot\|)$ be a 2-normed space. Then, for each $p \ge 1$, and each $\alpha_i \ge 0$, $i = 1,2,...,n$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and each $z, x_1,..., x_n \in L$, following holds true

$$\begin{aligned}&[\sum_{k=1}^{n} \|x_k,z\|^p - n^{1-p}\|\sum_{k=1}^{n} x_k,z\|^p] \max_{1\le i\le n}\{\alpha_i\} \ge \sum_{k=1}^{n} \alpha_k \|x_k,z\|^p - \|\sum_{k=1}^{n} \alpha_k x_k,z\|^p,\\ &\sum_{k=1}^{n} \alpha_k \|x_k,z\|^p - \|\sum_{k=1}^{n} \alpha_k x_k,z\|^p \ge [\sum_{k=1}^{n} \|x_k,z\|^p - n^{1-p}\|\sum_{k=1}^{n} x_k,z\|^p] \min_{1\le i\le n}\{\alpha_i\}.\end{aligned} \tag{27}$$

Proof. By Lemma 4, for each $z \in L$ and for each $p \ge 1$ the function $f_{p,z}: L \to \mathbf{R}$ defined by $f_{p,z}(x) = \|x,z\|^p$, $x \in L$ is convex. Now, the inequalities (27) are implied by Theorem 5, when used on the function $f_{p,z}$ for $p_i = \alpha_i$, $i = 1,2,...,n$ and $q_i = \frac{1}{n}$, $i = 1,2,...,n$.

Consequence 1. Let $(L,\|\cdot,\cdot\|)$ be a 2-normed space. Then for each $p \ge 1$ and for all $z, x_1,..., x_n \in L$, such that the sets $\{x_i, z\}, i = 1,2,...,n$ are linearly independent, the following holds true

$$\begin{aligned}&\sum_{i=1}^{n} \|x_i,z\|^p - n^{1-p}\|\sum_{i=1}^{n} x_i,z\|^p \ge [\sum_{i=1}^{n} \|x_i,z\|^{p-1} - (\sum_{i=1}^{n} \tfrac{1}{\|x_i,z\|})^{1-p} \|\sum_{i=1}^{n} \tfrac{x_i}{\|x_i,z\|},z\|^p] \min_{1\le i\le n}\{\|x_i,z\|\},\\ &[\sum_{i=1}^{n} \|x_i,z\|^{p-1} - (\sum_{i=1}^{n} \tfrac{1}{\|x_i,z\|})^{1-p} \|\sum_{i=1}^{n} \tfrac{x_i}{\|x_i,z\|},z\|^p] \max_{1\le i\le n}\{\|x_i,z\|\} \ge \sum_{i=1}^{n} \|x_i,z\|^p - n^{1-p}\|\sum_{i=1}^{n} x_i,z\|^p.\end{aligned} \tag{28}$$

Proof. If $p_i \ge 0, i = 1,...,n$, $\sum_{i=1}^{n} p_i > 0$ and $\alpha_i = p_i(\sum_{k=1}^{n} p_k)^{-1} \ge 0, i = 1,...,n$, then

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} p_i(\sum_{k=1}^{n} p_k)^{-1} = 1.$$

Hence, using (27), we get

$$[\sum_{i=1}^{n}\| x_i,z\|^p - n^{1-p}\|\sum_{i=1}^{n} x_i,z\|^p]\max_{1\le i\le n}\{p_i\} \ge \sum_{i=1}^{n} p_i\| x_i,z\|^p - (\sum_{i=1}^{n} p_i)^{1-p}\|\sum_{i=1}^{n} p_i x_i,z\|^p,$$
$$\sum_{i=1}^{n} p_i\| x_i,z\|^p - (\sum_{i=1}^{n} p_i)^{1-p}\|\sum_{i=1}^{n} p_i x_i,z\|^p \ge [\sum_{i=1}^{n}\| x_i,z\|^p - n^{1-p}\|\sum_{i=1}^{n} x_i\|^p,z]\min_{1\le i\le n}\{p_i\}. \tag{29}$$

Finally, when $p_i=\frac{1}{\|x_i,z\|}$, $i=1,2,...,n$, the inequalities (29) imply (28).

Consequence 2. Let $(L,\|\cdot,\cdot\|)$ be a 2-normed space. Then for all $z,x_1,...,x_n\in L$, such that the sets $\{x_i,z\}$, $i=1,2,...,n$ are linearly independent, the following inequalities are satisfied

$$[n-\|\sum_{j=1}^{n}\frac{x_j}{\|x_j,z\|},z\|]\max_{1\le i\le n}\{\| x_i,z\|\} \ge \sum_{j=1}^{n}\| x_j,z\| - \|\sum_{j=1}^{n} x_j,z\|,$$
$$\sum_{j=1}^{n}\| x_j,z\| - \|\sum_{j=1}^{n} x_j,z\| \ge [n-\|\sum_{j=1}^{n}\frac{x_j}{\|x_j,z\|},z\|]\min_{1\le i\le n}\{\| x_i,z\|\}. \tag{30}$$

Proof. The inequalities (30) follow by the inequalities (28), for $p=1$.

Consequence 3. Let $(L,\|\cdot,\cdot\|)$ be a 2-normed space. For all $x,y,z\in L$ such that the sets $\{x,z\}$ and $\{y,z\}$ are linearly independent, followings holds true

$$\| x+y,z\| \le \| x,z\| + \| y,z\| - (2-\|\frac{x}{\|x,z\|}+\frac{y}{\|y,z\|},z\|)\min\{\| x,z\|,\| y,z\|\}, \tag{31}$$

$$\| x+y,z\| \ge \| x,z\| + \| y,z\| - (2-\|\frac{x}{\|x,z\|}+\frac{y}{\|y,z\|},z\|)\max\{\| x,z\|,\| y,z\|\}. \tag{32}$$

Proof. The inequalities (30), when $n=2$ and $x_1=x, x_2=y$ directly imply the inequalities (31) and (32).

Consequence 4. Let $(L,\|\cdot,\cdot\|)$ be a 2-normed space. For all $x,y,z\in L$ such that the sets $\{x,z\}$ and $\{y,z\}$ are linearly independent, holds true following

$$\|\frac{x}{\|x,z\|}+\frac{y}{\|y,z\|},z\| \ge \frac{\|x+y,z\| - |\,\|x,z\|-\|y,z\|\,|}{\min\{\|x,z\|,\|y,z\|\}}, \tag{33}$$

$$\|\frac{x}{\|x,z\|}+\frac{y}{\|y,z\|},z\| \le \frac{\|x+y,z\| + |\,\|x,z\|-\|y,z\|\,|}{\max\{\|x,z\|,\|y,z\|\}}. \tag{34}$$

Proof. For all positive real numbers a and b it holds $2\min\{a,b\}-a-b=-|a-b|$, and further using the inequality (31) we get the following

$$\begin{aligned}\|\frac{x}{\|x,z\|}+\frac{y}{\|y,z\|},z\|\min\{\| x,z\|,\| y,z\|\} &\ge \| x+y,z\| + 2\min\{\| x,z\|,\| y,z\|\} - \| x,z\| - \| y,z\| \\ &= \| x+y,z\| - |\,\| x,z\| - \| y,z\|\,|,\end{aligned}$$

which is equivalent to the inequality (34).

In the theorem below we will give the necessary and sufficient condition for the inequality (31) and (32) to be transformed at equalities, when the space is strictly convex.

Theorem 7. Let $(L,\|\cdot,\cdot\|)$ be a strictly convex 2-normed space, and $x,y,z\in L$ are such that the sets $\{x,z\}$ and $\{y,z\}$ are linearly independent and $\| x,z\| < \| y,z\|$. Then

$$\| x+y,z\| + (2-\|\frac{x}{\|x,z\|}+\frac{y}{\|y,z\|},z\|)\| x,z\| = \| x,z\| + \| y,z\| \tag{35}$$

if and only if exists $\alpha\in(0,1)$ such that $x=\pm\alpha y$.

Proof. The parallelepiped inequality and $\| x,z \|<\| y,z \|$ imply

$$\begin{aligned} \| x+y,z \| &= \| \tfrac{\|x,z\|}{\|x,z\|}x + \tfrac{\|x,z\|}{\|y,z\|}y + (1-\tfrac{\|x,z\|}{\|y,z\|})y,z \| \\ &\leq \| x,z \| \cdot \| \tfrac{x}{\|x,z\|} + \tfrac{y}{\|y,z\|}, z \| + (1-\tfrac{\|x,z\|}{\|y,z\|}) \| y,z \| \\ &= \| x,z \| \cdot \| \tfrac{x}{\|x,z\|} + \tfrac{y}{\|y,z\|}, z \| + \| y,z \| - \| x,z \| \\ &= \| x,z \| + \| y,z \| - (2 - \| \tfrac{x}{\|x,z\|} + \tfrac{y}{\|y,z\|}, z \|) \| x,z \| . \end{aligned} \tag{36}$$

This means that the equality (35) holds if and only if the inequality (36) become an equality, i.e. if and only if

$$\| x + \tfrac{\|x,z\|}{\|y,z\|}y + (1-\tfrac{\|x,z\|}{\|y,z\|})y,z \| = \| x + \tfrac{\|x,z\|}{\|y,z\|}y,z \| + \| (1-\tfrac{\|x,z\|}{\|y,z\|})y,z \| . \tag{37}$$

But, L is strictly convex, and using Lemma2 we get that the equality (37) is equivalent to the following

$$\frac{x+\frac{\|x,z\|}{\|y,z\|}y}{\|x+\frac{\|x,z\|}{\|y,z\|}y,z\|} = \frac{(1-\frac{\|x,z\|}{\|y,z\|})y}{\|(1-\frac{\|x,z\|}{\|y,z\|})y,z\|}, \tag{38}$$

i.e. to the equality

$$x = (\| \tfrac{x}{\|x,z\|} + \tfrac{y}{\|y,z\|}, z \| - 1)\tfrac{\|x,z\|}{\|y,z\|} y .$$

Let

$$\alpha = (\| \tfrac{x}{\|x,z\|} + \tfrac{y}{\|y,z\|}, z \| - 1)\tfrac{\|x,z\|}{\|y,z\|} .$$

Then $x = \alpha y$. But, $\| x,z \| < \| y,z \|$, so $0 < |\alpha| < 1$.

Conversely, if $x = \alpha y$, for $0 < |\alpha| < 1$, then

$$\frac{x}{\|x,z\|} + \frac{y}{\|y,z\|} = (1+\frac{\alpha}{|\alpha|})\frac{y}{\|y,z\|} .$$

But, $1+\frac{\alpha}{|\alpha|} > 0$, so

$$\| \frac{x}{\|x,z\|} + \frac{y}{\|y,z\|}, z \| = 1 + \frac{\alpha}{|\alpha|} ,$$

i.e. the following is true

$$\frac{x}{\|x,z\|} + \frac{y}{\|y,z\|} = \| \frac{x}{\|x,z\|} + \frac{y}{\|y,z\|}, z \| \frac{y}{\|y,z\|} .$$

The last equality is equivalent to the equality (38) and to the equality (37). Therefore, in (36) apply the equality, i.e. is true the equality (35).

Remark 4. In the proof of Theorem 7, we actually proved the inequality (36). So, mentioned that $\min\{\| x,z \|, \| y,z \|\} = \| x,z \|$, in fact we proved the inequality (31) on some other way. Similarly, using facts that $\max\{\| x,z \|, \| y,z \|\} = \| y,z \|$ and

$$\begin{aligned} \| y,z \| \cdot \| \tfrac{x}{\|x,z\|} + \tfrac{y}{\|y,z\|}, z \| &= \| x + y + (\tfrac{\|y,z\|}{\|x,z\|} - 1)x, z \| \leq \| x+y,z \| + (\tfrac{\|y,z\|}{\|x,z\|} - 1) \| x,z \| \\ &= \| x+y,z \| + \| y,z \| - \| x,z \|, \end{aligned}$$

we get another proof of the inequality (32).

4. Conclusion

In our previous considerations we have already proved few inequalities at 2-normed space, which actually are generalizations of appropriate equalities at 2-normed space. Theorem 6 and Consequences 1 and 2 generalize the inequalities which hold true at normed space, and are already proved at [8], [9], [10], [11] and [12], and furthermore at [5] are proved applying the functional (25) and are corrected at [1]. Naturally, raises questions of generalization to other inequalities which hold true at normed space, and of generalization to previously reviewed inequalities at *n*-normed space.

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