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Fundamental Results in Dynamical Systems

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Abstract: *In this paper we take a review of the linear and Nonlinear systems of ordinary differential equations*

$$x' = Ax, \tag{1}$$

where $x \in R^n$, A *is an* $n \times n$ *matrix and* $x' = \frac{dx}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$

The solution of the linear system (1) together with the initial condition $x(0) = x_0$ *is given by* $x = e^{At}x_0$, *where* e^{At} *is an* $n \times n$ *matrix function defined by its Taylor series. In addition to this, we also discuss the nonlinear system of differential equation*

$$x' = f(x), \tag{2}$$

where $f : E \to R^n$, *E is an open subset of* R^n.

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1. Introduction

The method of separation of variables can be used to solve the first order linear differential equations $x' = ax$. The general solution is given by $x(t) = ce^{At}$ where the constant $c = x(0)$, the value of the function $x(t)$ at time $t = 0$.Now consider the uncoupled linear system $x_1' = -x_1$, $x_2' = 2x_2$.

This can be written as $x' = Ax$, where $A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$.

The solution of this system can be given by $x_1(t) = c_1e^{-t}$, $x_2(t) = c_2e^{2t}$ or equivalently by

$$x(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} c, \quad \text{where } c = x(0).$$

2. Fundamental Results for Linear System

Let A be an $n \times n$ matrix. In this section we discuss the fundamental fact that for $x_0 \in R^n$ the initial value problem $x' = Ax$, $x(0) = x_0$ has unique solution for all $t \in R$ which is given by $x(t) = x_0 e^{At}$. In order to prove this, we first compute the derivative of the exponential function e^{At} using the basic fact from analysis, that the two convergent limit processes can be interchanged if one of them converges uniformly.

Definitions: Let A be an $n \times n$ matrix . Then for $t \in R$, $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$.

Proposition 1: If S and *T* are linear transformations on R^n which commute then $e^{S+T} = e^S . e^T$.

Proof: If the transformations S and T are commuting, then we have *ST=TS.*

By the Binomial theorem

$$(S+T)^n = n! \sum_{j+k=n} \frac{S^j T^k}{j!k!}$$

$$\therefore \quad e^{S+T} = \sum_{n=0}^{\infty} \sum_{j+k=n} \frac{S^j T^k}{j!k!}$$

$$= \sum_{j=0}^{\infty} \frac{S^j}{j!} \sum_{k=0}^{\infty} \frac{T^k}{k!} = e^S e^T .$$

Setting S = -T in above proposition we obtain following Corollary

Corollary1: If T is a linear transformation on R^n, the inverse of linear transformation of e^T is given by $(e^T)^{-1} = e^{-T}$.

Lemma 1:- Let A be a square matrix, then $\frac{d}{dt} e^{At} = Ae^{At}$.

Proof:-Science A commute with itself, it follows from proposition 1 and definition

$$\frac{d}{dt} e^{At} = \lim_{h \to 0} \frac{e^{A(t+h)} - e^{At}}{h}$$

$$= \lim_{h \to 0} e^{At} \frac{(e^{Ah} - I)}{h}$$

$$= e^{At} \lim_{h \to 0} \lim_{k \to \infty} \left(A + \frac{A^2 h}{2!} + \frac{A^3 h^2}{3!} + \ldots + \frac{A^k h^{k-1}}{k!} \right)$$

$$= Ae^{At} .$$

Since e^{Ah} converges uniformly for $|h| \le 1$, we can interchange the two limits. [6]

Theorem 1: (The fundamental theorem for linear system)

Let A be an $n \times n$ matrix. Then for given $x_0 \in R^n$ the initial value problem $x' = Ax$, $x(0) = x_0$ has a unique solution given by $x(t) = e^{At} x_0$.

Proof: By the preceding Lemma, If $x(t) = e^{At} x_0$, then $x'(t) = \frac{d}{dt} e^{At} x_0 = Ae^{At} x_0 = Ax(t)$ for all $t \in R$. Also $x(0) = Ix_0 = x_0$. Thus $x(t) = e^{At} x_0$ is a solution.

Uniqueness of the solution To see that this is the only solution, let $x(t)$be any solution of the initial value problem (1) and set $y(t)=e^{-At}x(t)$. Then from the above lemma and the fact that $x(t)$is a solution of (1), it follows that

$$y'(t)=-Ae^{-At}x(t)+e^{-At}x'(t)$$

=0 for all $t\in R$ since e^{-At} and A commute. Thus $y(t)$ is a constant. Setting $t=0$ shows that $y(t)=x_0$ and therefore any solution of the initial value problem is given by

$x(t)=e^{At}y(t)=e^{At}x_0$. Thus the result.

3. Fundamental Results for Nonlinear System

Before starting and proving the fundamental results for the nonlinear system (2), we discuss the basic terminology and notations concerning the derivative Df of a function $f:R^n\rightarrow R^n$.

Definition: The function $f:R^n\rightarrow R^n$ is differentiable at $x_0\in R^n$ if there is a linear transformation $Df\in L(R^n)$ that satisfies

$$\lim_{|h|\mapsto 0}\frac{|f(x_0+h)-f(x_0)-Df(x_0)h|}{|h|}=0$$

the linear transformation $Df(x_0)$ is called the derivative of f at x_0.

The following result established by [6] gives us a method for computing the derivative in coordinates .

Theorem 2: If $f:R^n\rightarrow R^n$ is differentiable at x_0, then the partial derivatives $\frac{\partial f_i}{\partial x_j}$,i,j=1,2,…n, all exist at x_0 and for all $x\in R^n$, $Df(x_0)x=\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}(x_0)x_j$.

Proof: Ref [6].

Thus if f is differentiable function , the derivative Df is given by the $n\times n$ Jacobian matrix $Df=\left[\frac{\partial f_i}{dx_j}\right]$, it is assumed that the function $f(x)$ is continuously differentiable ; i.e. that the derivative $Df(x)$ is considered as a mapping $Df:R^n\rightarrow L(R^n)$ and is a continuous function of x in some open subset $E\subset R^n$. The linear spaces R^n and $L(R^n)$ are endowed with the Euclidean norm |.| and the operator norm ||.||.

Suppose that E is an open Subset of R^n, the higher order derivatives $D^k f(x_0)$ of a function $f:E\rightarrow R^n$ are defined in similar way and it can be shown that $f\in C^k(E)$ if and only if the partial derivatives $\frac{\partial^k f_i}{\partial x_{j_1}\partial x_{j_2}\ldots\partial x_{j_k}}$ with $i,j_1,j_2,\ldots,j_k$=1,…, n exist and are continuous on E. Furthermore, $D^2f(x_0):E\times E\rightarrow R^n$ and for $(x,y)\in E\times E$ we have $D^2f(x_0)(x,y)=\sum_{j_1,j_2=1}^{n}\frac{\partial^2 f(x_0)}{\partial x_{j_1}\partial x_{j_2}}x_{j_1}y_{j_2}$. Similar results hold for higher ordered derivatives .

Definition: Suppose that $f\in C(E)$, where E is an open subset of R^n. Then $x(t)$ is a solution of the differential equation (2)on an interval I if it is differentiable on I and for all

$t \in I$, $x(t) \in E$ and $x'(t) = f(x(t))$.Also, for a given $x_0 \in E$, *x(t)* is a solution of the initial value problem $x' = f(x)$, $x(t_0) = x_0$ on an interval I if $t_0 \in I$, $x(t_0) = x_0$ is a solution of the differential equation on the interval *I*.

Definition: Let E be an open subset of R^n. A function $f : E \to R^n$ is said to be locally Lipschitz on E if for each $x_0 \in E$ there is an ε-neighborhood of x_0, $N_\varepsilon(x_0) \subset E$ and a constant *K > 0* such that for all $x, y \in N_\varepsilon(x_0) \subset E$,

$$| f(x) - f(y) | \le K | x - y |.$$

If $| f(x) - f(y) | \le K | x - y |$ holds for all $x, y \in E$ then it is Lipschitz on *E*.

Lemma2: Let *E* be an open subset of R^n and let $f : E \to R^n$.If $f \in C(E)$, *f* is locally Lipschitz on *E*.

Proof: Refer [8]

Let $I = [-a, a]$ the norm on C(I) is define as $\| u \| = \sup_I | u(t) |$. Convergence in this norm is equivalent to uniform convergence.

Definition: Let V be a normed linear space . Then a sequence $\{u_k\} \subset V$ is called a Cauchy sequence if for all $\varepsilon > 0$ there is a positive integer N such that $k, m \ge N$ implies that $\| u_k - u_m \| < \varepsilon$.

The space V is called complete if every Cauchy sequence in V converges to an element in V.

Theorem3: For $I = [-a, a]$, C(I) is a complete normed linear space. [8]

Theorem4: (Fundamental Existence theorem). Let E be an open subset of R^n containing x_0 and assume that $f \in C(I)$. Then there exists an *a > 0* such that the initial value problem $x' = f(x)$, $x(0) = x_0$ has a unique solution *x(t)* on the interval $[-a, a]$.

Proof: Since $f \in C(I)$, it follows from the lemma that there is an ε-neighborhood $N_\varepsilon(x_0) \subset E$ and a constant *K > 0* such that for all $x, y \in N_\varepsilon(x_0)$,

$$| f(x) - f(y) | \le K | x - y |.$$

Let $b = \varepsilon/2$. Then the continuous function $f(x)$ is bounded on the compact set $N_0 = \{x \in R^n / |x - x_0| \le b\}$.

Let $M = \max_{x \in N_0} | f(x) |$.Let the successive approximations $u_k(t)$ be defined by the sequence of functions $u_0(t) = x_0$,

$$u_{k+1}(t) = x_0 + \int_0^t f(u_k(s))ds, \text{ for } k = 0,1,2,... \tag{1}$$

Then assuming that there exist an *a>0* such that $u_k(t)$ is defined and continuous on [-*a*, *a*] and satisfies

$$\max_{[-a,a]} |u_k(t) - x_0| \le b \tag{2}$$

It follows that $f(u_k(t))$ is defined and continuous on [-*a*, *a*] and therefore that

$$u_{k+1}(t) = x_0 + \int_0^t f(u_k(s))ds$$

is defined and continuous on [-*a*, *a*] and satisfies

$$|u_{k+1}(t) - x_0| \leq \int_0^t |f(u_k(s))|ds \leq Ma \text{ for all } t \in [-a, a].$$

Thus, choosing $0 < a \leq b/M$, it follows by induction that $u_k(t)$ is defined and continuous and satisfies (2) for all $t \in [-a, a]$ and k=1,2,….

Next since for all $t \in [-a, a]$ and for k=1,2,3,….,$u_k(t) \in N_0$, it follows from Lipschitz condition satisfied by *f* that for all $t \in [-a, a]$

$$|u_2(t) - u_1(t)| \leq \int_0^t |f(u_1(s)) - f(u_0(s))|ds$$

$$\leq K \int_0^t |u_1(s) - u_0(s)|ds$$

$$\leq Ka \max_{[-a,a]} |u_1(t) - x_0| \leq Kab$$

And assuming that

$$\max_{[-a,a]} |u_j(t) - u_{j-1}(t)| \leq (Ka)^{j-1} b \tag{3}$$

For some integer $j \geq 2$, it follows that for all $t \in [-a, a]$

$$|u_{j+1}(t) - u_j(t)| \leq \int_0^t |f(u_{j+1}(s)) - f(u_j(t))|ds$$

$$\leq K \int_0^t |u_j(s) - u_{j-1}(s)|ds$$

$$\leq Ka \max_{[-a,a]} |u_j(t) - u_{j-1}(t)$$

$$\leq (Ka)^j b$$

Thus ,it follows by induction that (3)holds for *j*=2,3,….Setting $\alpha = Ka$ and choosing $0 < a < 1/K$,we see that for $m > k \geq N$ and $t \in [-a, a]$

$$|u_m(t) - u_k(t)| \leq \sum_{j=k}^{m-1} |u_{j+1}(t) - u_j(t)|$$

$$\leq \sum_{j=N}^{\infty} |u_{j+1}(t) - u_j(t)|$$

$$\leq \sum_{j=N}^{\infty} \alpha^j b = \frac{\alpha^N}{1-\alpha} b.$$

This last quantity approaches zero as $N \to \infty$. Therefore, for all $\varepsilon > 0$ there exist an N such that $m, k \geq N$ implies that $\| u_m - u_k \| = \max_{[-a,a]} |u_m(t) - u_k(t)| < \varepsilon$;i.e. $\{u_k\}$ is a Cauchy sequence of continuous functions in $C([-a,a])$. Therefore this sequence converges to the continuous function $u(t)$ uniformly for all $t \in [-a,a]$ as $k \to \infty$. Taking limit of both sides of equation (1) defining the successive approximations , we see that the continuous function

$$u(t) = \lim_{k\to\infty} u_k(t) \tag{4}$$

satisfies the integral equation

$$u(t) = x_0 + \int_0^t f(u(s))ds \tag{5}$$

for all $t \in [-a,a]$. We have used the fact that the integral and the limit can be interchanged since the limit in (4) is uniform for all $t \in [-a,a]$.

Also, since $u(t)$is continuous, $f(u(t))$ is continuous and by fundamental theorem of calculus, the right hand of the integral equation (5) is differentiable and $u'(t) = f(u(t))$ for all $t \in [-a,a]$. Furthermore, $u(0) = x_0$ and from (1), it follows that $u(t) \in N_\varepsilon(x_0) \subset E$ for all $t \in [-a,a]$. Thus $u(t)$ is a solution of initial value problem defined on $[-a,a]$.

Uniqueness of Solution Let $u(t)$ and $v(t)$ be two solutions of this initial value problem . Then the continuous function $|u(t)-v(t)|$ achieves its maximum at some point $t_1 \in [-a,a]$. It follows that

$$\begin{aligned}
\|u-v\| &= \max_{[-a,a]} |u(t)-v(t)| \\
&= \left| \int_0^{t_1} f(u(s)) - f(v(s))ds \right| \\
&\le \int_0^{|t_1|} |f(u(s)) - f(v(s))|ds \\
&\le K \int_0^{|t_1|} |u(s)-v(s)|ds \\
&\le Ka \max_{[-a,a]} |u(t)-v(t)| \\
&\le Ka\|u-v\|.
\end{aligned}$$

But $Ka < 1$ and this last inequality can only be satisfies if $\|u-v\| = 0$

Thus $u(t) = v(t)$ on $[-a,a]$ i.e. the successive approximations defined by (1) converges uniformly to the unique solution of the initial value problem on $[-a,a]$ where a is any number satisfying

$$0 < a < \min\left(\frac{b}{M}, \frac{1}{K}\right).$$

Remark: Using this result we can prove following theorem in similar way.

Theorem5: If the matrix value function $A(t)$ is continuous on $[-a_0, a_0]$ then there exists an $a > 0$ such that the initial value problem $\phi' = A(t)\phi$, $\phi(0) = I$, where I is an identity matrix of order $n \times n$, has unique solution $\phi(t)$ on $[-a,a]$.

Proof Define $\phi_0(t)$ =I and $\phi_{k+1}(t) = I + \int_0^t A(s)\phi_k(s)ds$. As the continuous matrix function $A(t)$ is bounded i.e.it satisfies $\|A(t)\| \le M_0$ for set of all t in the compact set $[-a_0, a_0]$. Hence using above techniques, the successive approximations $\phi_k(t)$ converges to $\phi(t)$ on some interval $[-a, a]$ with $a < 1/M_0$ and $a \le a_0$.

4. CONCLUSION

Let A be an $n \times n$ matrix. The fundamental fact that for $x_0 \in R^n$, the initial value problem $x' = Ax$, $x(0) = x_0$ has unique solution for all $t \in R$ which is given by $x(t) = x_0 e^{At}$.

The existence and uniqueness results are the basic fundamental results for both linear and nonlinear continuous Dynamical systems.

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