International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 2, Issue 5, May 2014, PP 441-442
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Another Look at Planetary Motion

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Abstract: *In this brief note, the classical and relativistic equations of planetary motion are studied using polar coordinates in a different way than the standard approach. Conditions are then given when the relativistic equations yield bounded solutions.*

Keywords: *planetary,bounded, dynamicalsystem, differential equations, coordinates*

1. Introduction

In this note, a straightforward account will be given of the well-known differential equations of planetary motion. However, we will discuss them using the (r,θ)coordinatesin a slightly different way than the usual case (see [1, pp. 231-235] for the classicalapproach and [2, pp.471-496] for a thorough presentation of the behavior of a body in motion subject to a central force). This discussion simplifies the analysis of these fundamental equations of celestial mechanics (see [3] for an excellent introduction to this field).

2. Methods

The first equation is the classical Newtonian model while the second one is derived from general relativity (see [4, pp. 270-276] for further details). The two equations are:

(1) $u''(\theta) + u(\theta) = c$(where $u=1/r$, r being the radius from the given object to one of the foci which is the sun for the purpose of this discussion and c is a certain positive constant) and

(2) $u''(\theta) + u(\theta) - c_1 u(\theta)^2 = c_2$(where c_1 and c_2 are other positive constants).

In our discussion, we shall solve the first equation exactly and the second one implicitly by treating it as a two dimensional system. This will give us some clarity into the behavior of their solutions.

3. Main Results and Discussion

We first begin our analysis by solving equation (1) which has the general solution $u=c + a\sin(\theta) + b\cos(\theta)$ where a and b are constants. Noting that $u=1/r$ we may therefore conclude that $r=(c+ a\sin(\theta) + b\cos(\theta))^{-1}$ which is the general form of an ellipse, the planetary orbit. This general solution covers all cases including the casewhen the major and minor axes of the ellipse are not parallel or perpendicular to the XY axes.When setting a=0, we have theequation of theellipse in standard form$r=1/(c+b\cos(\theta))$. The only constraint is $c+b\cos(\theta) \neq 0$.

As far as equation (2) is concerned we can show directly that all solutions are bounded when given certain initial conditions. First, multiply (2) by 2u' and integrate for 0 to θ obtaining

(3) $u'(\theta)^2 + u(\theta)^2 - 2c_1u(\theta)^3/3 = 2c_2\, u(\theta) - 2c_2\, u(0) + u'(0)^2 + u(0)^2 - 2c_1u(0)^3/3$.

Next, using the fact that $u=1/r$ and $u' = -r'/ r^2$and then multiplying equation (3) by r^4 transforms equation (3) into

(4) $r'(\theta)^2+r(\theta)^2-2c_1r(\theta)/3=2c_2r(\theta)^3-2c_2r(\theta)^4u(0)+kr(\theta)^4$

where $k= u'(0)^2 + u(0)^2 - 2c_1u(0)^3/3$. If $k-2c_2u(0) < 0$, then should $r \to \infty$ the LHS of (4) approaches ∞ while the RHS approaches -∞ which is impossible. In other words, the solutions must remain bounded as $t \to \infty$ given these conditions.Should $k-2c_2u(0) \geq 0$, then the solutions may be unbounded (note: the equilibrium points of (2) in the (u, θ) plane correspond to regions of unboundedness in the (r,θ) plane).

We could also look at equation (2) another way by transforming it into the following dynamical system

(5) $u' = v$

$v' = -u + c_1u^2 + c_2.$

Then, we convert (5) into the first order differential equation,

(6) $dv/du = (-u + c_1u^2 + c)/v$. This becomes

(7) $v\,dv = (-u + c_1u^2 + c_2)\,du$.

Finally, integrating equation (7) from 0 to θ yields the following result

(8) $½(v(\theta)^2 - v(0)^2) = -u(\theta)^2/2 + c_1u(\theta)^3/3 + c_2u(\theta) + u(0)^2/2 - c_1u(0)^3/3 - c_2u(0)$.

Letting $u = 1/r$ and $u' = v = -r'/r^2$ again yield equation (4) after some algebraic manipulation.

4. Conclusion

Using standard methods from differential equations, the above analysis clearly gives a straightforward and precise analysis of planetary motionboth classical and relativistic which plays an important role in celestial mechanics.

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Author's Biography

I am currently an adjunct associate professor in the Department of Management and Finance at the UniversityofMaryland University College. Previously, I was a forty year federal civil servant with the Social Security Administration where I held various professional positions including mathematical statistician and information technology specialist.I am listed in the Who's Who in America, World Directoryof Mathematicians and am a member of the Mathematical Association of America. Finally, I am the author of over thirty mathematical papers and articles mostly in differential equations which is my specialty.