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Approximation by Some Linear Positive Operators in Lp Spaces

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Abstract: *The summation integral type Baskakov operator was introduced by Gupta and Srivastava [4] in 1993. In the present paper, we extend our studies and introduce the Baskakov- Szasz Stancu operators. We prove a direct theorem for the linear combinations of Baskakov- Szasz Stancu type operators. To prove our main theorem, we use the technique of linear approximation method viz. Steklov mean.*

Keywords: *Steklov mean, linear combinations, linear positive operators, Stancu operators*

Ams subject classification: *41A25, 41A35*

1. Introduction

For $f \in Lp[0, \infty)$, $p \geq 1$, modified Baskakov - Szasz operators defined by Gupta and Srivastava [4] in 1993 are as

$$S_n(f, x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} q_{n,k}(t) f(t)dt, \quad x \in [0, \infty), \tag{1}$$

In 1983, the Stancu type generalization of Bernstein operators was given in [9]. In 2010 in [1], the authors have studied the Stancu type generalization of the q-analogue of classical Baskakov operators. In the recent years for similar type of operators some approximation properties have been discussed by Maheshwari [7], Maheshwari-Sharma [8] etc. Motivated by the recent work on Stancu type operators, here we propose the Stancu type generalization of Baskakov-Szasz Stancu operators, for $0 \leq \alpha \leq \beta$ as

$$S_{n,\alpha,\beta}(f, x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} q_{n,k}(t) f\left(\frac{nt+\alpha}{n+\beta}\right)dt, \quad x \in [0, \infty), \tag{2}$$

where

$$p_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}, \qquad q_{n,k}(t) = \frac{e^{-nt}(nt)^k}{k!}. \tag{3}$$

On putting $\alpha = \beta = 0$ operators (2) reduce to operators defined in (1). It is observed that the order of approximation for these operators is $O(n^{-1})$. To improve the order of approximation, we consider the linear combinations of these operators $S_{n,\alpha,\beta}(f, k, x)$ of the operators $S_{d_j n,\alpha,\beta}(f, x)$ as

$$S_{n,\alpha,\beta}(f, k, x) = \sum_{j=0}^{k} C(j, k) S_{d_j n,\alpha,\beta}(f, x), \tag{4}$$

where

$$C(j,k)=\prod_{\substack{i=0\\ i\neq j}}^{k}\frac{d_j i}{d_j-d_i},k\neq 0 \quad and \quad C(0,0)=1. \tag{5}$$

We can write the operators (2) as

$$S_{n,\alpha,\beta}(f,x)=\int_0^\infty W(n,x,t)f\left(\frac{nt+\alpha}{n+\beta}\right)dt, \tag{6}$$

where

$$W(n,x,t)=n\sum_{k=0}^{\infty}p_{n,k}(x)q_{n,k}(t).$$

The k^{th} linear combinations $S_n(f,k,x)$, considered by May [6] for the operators $S_{d_j n,\alpha,\beta}(f,x)$ are defined by

$$S_n(f,k,x)=\begin{vmatrix}1 & d_0^{-1} & \dots & d_0^{-k}\\ 1 & d_1^{-1} & \dots & d_1^{-k}\\ \dots\dots\dots & \dots\dots\dots & \dots\dots & \\ 1 & d_k^{-1} & \dots & d_k^{-k}\end{vmatrix}^{-1}\times\begin{vmatrix}S_{d_0 n}(f,x) & d_0^{-1} & \dots & d_0^{-k}\\ S_{d_1 n}(f,x) & d_1^{-1} & \dots & d_1^{-k}\\ & \dots\dots\dots & \dots & \\ S_{d_k n}(f,x) & d_k^{-1} & \dots & d_k^{-k}\end{vmatrix}, \tag{7}$$

where $d_0,d_1,d_2,\dots,d_k$ are $(k+1)$ arbitrary but fixed distinct positive numbers. In this paper we have considered $0<a_1<a_2<a_3<b_3<b_2<b_1<\infty, 0<a<b<\infty$ and $I_i=[a_i,b_i], i=1,2,3$. We denote by H. For $f\in Lp[0,\infty)$ and $1\leq p<\infty$, the Steklov mean $f_{\eta,m}$ of m^{th} order corresponding to f is defined by

$$f_{\eta,m}(t)=\eta^{-m}\int_{-\eta/2}^{\eta/2}\int_{-\eta/2}^{\eta/2}\dots\dots\int_{-\eta/2}^{\eta/2}[f(t)+(-1)^{m-1}\Delta_h^m f(t)dt_1,dt_2,\dots,dt_m],$$

where $t=\sum_{i=1}^{m}u_i$ and $\Delta_h^m f(t)$ is the m^{th} order forward difference of function f with step length h, defined as

$$\Delta_h^m f(t)=\Delta_h^{m-1}\Delta_h^1 f(t)=\Delta_h^{m-1}[f(t+h)-f(t)].$$

From [10, 5] we have

(1) $f_{\eta,m}$ has derivative up to order m, $f_{\eta,m}^{(m-1)}\in AC(I_1)$, and $f_{\eta,m}^{(m-1)}$ exists a.e and belong to $Lp(I_1)$;

(2) $\left\|f_{\eta,m}^{(r)}\right\|_{L_p(I_2)}\leq H\eta^{-r}\omega_r(f,\eta,p,I_1), r=1,2,\dots\dots m;$

(3) $\left\|f-f_{\eta,m}\right\|_{L_p(I_2)}\leq H\omega_m(f,\eta,p,I_1);$

(4) $\left\|f_{\eta,m}\right\|_{L_p(I_2)}\leq H\left\|f\right\|_{L_p(I_1)};$

(5) $\left\|f_{\eta,m}^{(r)}\right\|_{L_p(I_2)}\leq H\eta^{-m}\left\|f\right\|_{L_p(I_1)}, \quad r=1,2,\dots, \quad m;$

Here we represent absolute continuous function on $[a,b]$ as $AC[a,b]$ and H are certain constants defined on I, but are independent of f and n. $BV[a,b]$ denotes the set of all functions of bounded variation on $[a,b]$. The semi norm $\|f\|_{BV[a,b]}$ is defined by the total variation of f on $[a,b]$. For $f \in L_p[a,b], 1 < p < \infty$, the Hardy-Little wood majorant of f is defined as

$$h_f(x) = \sup_{\xi \to x} \frac{1}{\xi - x} \int_x^{\xi} f(t)\,dt, \quad (a \le \xi \le b).$$

In the present work we establish some direct results on Lp -norm for the linear combination of the Baskakov- Szasz- Stancu operators.

2. Moments Estimation and Auxiliary Results

In this section we estimate moments and mention certain basic results.

Lemma 1 ..*[4] Let the m^{th} order moment be defined by*

$$T_{n,\alpha,\beta,} = n \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} q_{n,k}(t) \left(\frac{nt + \alpha}{n + \beta} - x \right)^m dt, \tag{8}$$

then $T_{n,0}(x) = 1, \qquad T_{n,1}(x) = \dfrac{1 + \alpha - \beta x}{n + \beta}$

and $T_{n,2}(x) = \dfrac{x^2(n + \beta^2 + 2n\beta)}{(n+\beta)^2} + \dfrac{2x(n - \alpha\beta - n\beta - \beta)}{(n+\beta)^2} + \dfrac{2(1+\alpha) + \alpha^2}{(n+\beta)^2}$

and we have the recurrence relation $m - 1 \in N$

$$nT_{n,m+1}(x) = x(1+x)T'_{n,m}(x) + (m+1)T_{n,m}(x) + m\left[x(1+x) - \left(\frac{\alpha}{n+\beta} - x \right) \right] T_{n,m-1}(x) \tag{9}$$

Consequently for $x \ge 0$,

$$T_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right), \tag{10}$$

where $[\alpha]$ denotes the integral part of α. By using Holder's inequality we get the conclusion, for every fixed $x \in [0, \infty)$.

$$S_{n,\alpha,\beta,}\left(|t - x|^r, x\right) = O\left(n^{-r/2}\right), \quad \forall r > 0 \tag{11}$$

Lemma 2. *For $p \in N$ and n sufficiently large there hold,*

$$S_{n,\alpha,\beta,}\left[(t-x)^p, k, x\right] = n^{-(k+1)}\{Q(p,k,x) + o(1)\}, \quad t \in [0, \infty)$$

where $Q(p,k,x)$ are certain polynomials in x of degree $p/2$.

Proof of above Lemma is easy and can be seen on similar type of operators.

Lemma 3. *[3] Let $1 \le p \le \infty$, $f \in L_p[a,b]$, $f^{(k)} \in AC[a,b]$ and $f^{(k+1)} \in Lp[a,b]$, then*

$$\left\| f^{(j)} \right\|_{L_p[a,b]} \le H\left(\left\| f^{(k+1)} \right\|_{L_p[a,b]} + \|f\|_{L_p[a,b]} \right)$$

where $j = 1,2,..., k$, and H are certain constants depending only on j, k, p, a, b.

Lemma 4 . *[2] There exist the polynomials $q_{i,j,r}(x)$ on $[0, \infty)$, independent of n and k such* *that*

$$x^r(1+x)^r \frac{d^r}{dx^r} p_{n,k}(x) = \sum_{\substack{2i+j\le r \\ i,j\ge 0}}^{k} n^i (k-nx)^j q_{i,j,r}(x) p_{n,k}(x).$$

3. Direct Estimates

Theorem 1. *Let* $f \in L_p[0,\infty)$, $p>1$. *If f has* $(2k+2)$ *derivative on* I_1 *with* $f^{(2k+1)} \in AC(I_1)$ *then for* n *sufficiently large*

$$\left\| S_{n,\alpha,\beta}(f,k,.) - f \right\|_{L_p(I_2)} \le Hn^{-(k+1)}\left(\left\| f^{(k+1)} \right\|_{L_p(I_2)} + \left\| f \right\|_{L_p[0,\infty)} \right),$$

where the constant H is independent of n and f

Proof: By our assumptions, for $x \in I_2$ and $t \in I_1$, we have

$$f\left(\frac{nt+\alpha}{n+\beta}\right) = \sum_{j=0}^{2k+1} \frac{\left(\frac{nt+\alpha}{n+\beta}-x\right)^j}{j!} f^{(j)}(x) + \frac{1}{(2k+1)!}\int_x^t \left(\frac{nt+\alpha}{n+\beta}-u\right)^{2k+1} f^{(2k+2)}(u)\,du$$
$$+ F\left(\frac{nt+\alpha}{n+\beta},x\right)\left(1-\phi\left(\frac{nt+\alpha}{n+\beta}\right)\right), \tag{12}$$

where $\phi(t)$ denotes the characteristic function on I_1.

$$F\left(\frac{nt+\alpha}{n+\beta},x\right) = f\left(\frac{nt+\alpha}{n+\beta}\right) - \sum_{j=0}^{2k+1} \frac{\left(\frac{nt+\alpha}{n+\beta}-x\right)^j}{j!} f^{(j)}(x).$$

For all $t \in [0,\infty)$ and $x \in I_2$. Using (12) in (4), we have

$$S_{n,\alpha,\beta}(f,k,x) - f(x) = \sum_{j=1}^{2k+1} \frac{f^{(j)}(x)}{j!} S_{n,\alpha,\beta}\left(\left(\frac{nt+\alpha}{n+\beta}-x\right)^j,k,x\right)$$
$$+ \frac{1}{(2k+1)!} S_{n,\alpha,\beta}\phi\left(\frac{nt+\alpha}{n+\beta}\right)\int_x^t \left(\left(\frac{nt+\alpha}{n+\beta}-u\right)^{2k+1} f^{(2k+2)}(u)\,du,k,x\right)$$
$$+ S_{n,\alpha,\beta}\left(F\left(\frac{nt+\alpha}{n+\beta},x\right)\left(1-\phi\left(\frac{nt+\alpha}{n+\beta}\right),k,x\right)\right) =: \psi_1 + \psi_2 + \psi_3.$$

According to Lemma 2 and [3]

$$\left\|\psi_1\right\|_{L_p(I_2)} \le Hn^{-(k+1)}\left(\sum_{j=1}^{2k+1}\left\| f^{(j)} \right\|_{L_p(I_2)}\right)$$
$$\le Hn^{-(k+1)}\left(\left\| f^{(j)} \right\|_{L_p(I_2)} + \left\| f^{(2k+2)} \right\|_{L_p(I_2)}\right).$$

For finding I_2, let h_f be the Hardy-Littlewood Majorant [11] of $f^{(2k+2)}$ on I_1. Now using Holder's inequality (11), we obtain

$$R_1 = \left| S_{n,\alpha,\beta}\left(\phi\left(\frac{nt+\alpha}{n+\beta}\right) \int_x^t \left(\frac{nt+\alpha}{n+\beta} - u \right)^{2k+1} f^{(2k+2)}(u)\,du, x \right) \right|$$

$$\le S_{n,\alpha,\beta}\left(\phi\left(\frac{nt+\alpha}{n+\beta}\right) \int_t^x \left(\left|\left(\frac{nt+\alpha}{n+\beta} - u \right)^{2k+1}\right| \left| f^{2k+2}(u)\right| \left|du\right|, x \right) \right)$$

$$\le S_{n,\alpha,\beta}\left\{ \phi\left(\frac{nt+\alpha}{n+\beta}\right) \left|\left(\frac{nt+\alpha}{n+\beta} - x \right)\right|^{2k+1} \int_x^t \left| f^{2k+2}(u)\right| \left|du\right|, x \right\}$$

$$\le S_{n,\alpha,\beta}\left\{ \phi\left(\frac{nt+\alpha}{n+\beta}\right) \left(\frac{nt+\alpha}{n+\beta} - x \right)^{2k+2} \left| h_f\left(\frac{nt+\alpha}{n+\beta}\right)\right|, x \right\}$$

$$\le \left\{ S_{n,\alpha,\beta}\left(\left|\left(\frac{nt+\alpha}{n+\beta} - x \right)\right|^{(2k+2)q} \phi\left(\frac{nt+\alpha}{n+\beta}\right), x \right) \right\}^{1/q} \left\{ S_{n,\alpha,\beta}\left(\left| h_f\left(\frac{nt+\alpha}{n+\beta}\right)\right|^p \phi\left(\frac{nt+\alpha}{n+\beta}\right), x \right) \right\}^{1/p}$$

$$\le Hn^{-(k+1)} \left\{ S_{n,\alpha,\beta}\left(\left| h_f\left(\frac{nt+\alpha}{n+\beta}\right)\right|^p \phi\left(\frac{nt+\alpha}{n+\beta}\right), x \right) \right\}^{1/p}$$

$$\le Hn^{-(k+1)} \left[\int_{a_1}^{b_1} W(n,x,t) \left| h_f\left(\frac{nt+\alpha}{n+\beta}\right)\right|^p dt \right]^{1/p}.$$

Using Fubini's theorem and [12], we get

$$\left\| R_1 \right\|_{L_p(I_2)}^p \le Hn^{-(k+1)p} \int_{a_2}^{b_2} \int_{a_1}^{b_1} W(n,x,t) \left| h_f\left(\frac{nt+\alpha}{n+\beta}\right)\right|^p dt\,dx$$

$$\le Hn^{-(k+1)p} \int_{a_1}^{b_1} \left[\int_{a_2}^{b_2} W(n,x,t)\,dx \right] \left| h_f\left(\frac{nt+\alpha}{n+\beta}\right)\right|^p dt$$

$$\le Hn^{-(k+1)p} \int_{a_1}^{b_1} \left| h_f\left(\frac{nt+\alpha}{n+\beta}\right)\right|^p dt$$

$$\le Hn^{-(k+1)p} \left\| h_f\left(\frac{nt+\alpha}{n+\beta}\right)\right\|_{L_p(I_1)}^p$$

$$\le Hn^{-(k+1)p} \left\| f^{(2k+2)} \right\|_{L_p(I_1)}^p.$$

Hence,

$$\left\| R_1 \right\|_{L_p(I_2)} \le Hn^{-(k+1)p} \left\| f^{(2k+2)} \right\|_{L_p(I_1)}.$$

Consequently,

$$\left\| \psi_2 \right\|_{L_p(I_2)} \le Hn^{-(k+1)p} \left\| f^{(2k+2)} \right\|_{L_p(I_1)}.$$

For $t \in [0, \infty) \setminus [a_1, b_1]$, $x \in I_2$, $\exists \ \delta > 0$ such that $|t - x| \geq \delta$.

Thus

$$\left| S_{n,\alpha,\beta}\left(F\left(\frac{nt+\alpha}{n+\beta}, x\right)\left(1 - \phi\left(\frac{nt+\alpha}{n+\beta}\right), x\right)\right)\right| \leq \delta^{-(2k+2)} S_{n,\alpha,\beta}\left(\left| F\left(\frac{nt+\alpha}{n+\beta}, x\right)\right| \left(\frac{nt+\alpha}{n+\beta} - x\right)^{2k+2}, x\right)$$

$$\leq \delta^{-(2k+2)} S_{n,\alpha,\beta}\left(\left| f\left(\frac{nt+\alpha}{n+\beta}\right)\right| + \sum_{j=0}^{2k+1} \frac{\left|\left(\frac{nt+\alpha}{n+\beta} - x\right)\right|^j}{j!} \left| f^{(j)}(x)\right| \left(\frac{nt+\alpha}{n+\beta} - x\right)^{2k+2}, x\right)$$

$$\leq \delta^{-(2k+2)} [S_{n,\alpha,\beta}\left(\left| f\left(\frac{nt+\alpha}{n+\beta}\right)\right| \left(\frac{nt+\alpha}{n+\beta} - x\right)^{2k+2}, x\right)$$

$$+ \sum_{j=1}^{2k+1} \frac{\left| f^{(j)}(x)\right|}{j!} S_{n,\alpha,\beta}\left(\left|\left(\frac{nt+\alpha}{n+\beta} - x\right)\right|^{2k+2+j}, x\right)] = R_2 + R_3.$$

Using Holder's inequality and (11), we get

$$|R_2| \leq \delta^{-(2k+2)} S_{n,\alpha,\beta}\left(|f(x)|^p, x\right)^{1/p} S_{n,\alpha,\beta}\left(\left|\left(\frac{nt+\alpha}{n+\beta} - x\right)\right|^{(2k+2)q}, x\right)^{1/q}$$

$$\leq Hn^{-(k+1)} S_{n,\alpha,\beta}\left(\left| f\left(\frac{nt+\alpha}{n+\beta}\right)\right|^p, x\right)^{1/p}.$$

Again applying Fubini's theorem, we get

$$\int_{a_2}^{b_2} |R_2|^p \, dt \leq Hn^{-(k+1)p} \int_{a_2}^{b_2}\int_0^{\infty} W(n,x,t) \left| f\left(\frac{nt+\alpha}{n+\beta}\right)\right|^p dtdx$$

$$\leq Hn^{-(k+1)} \| f \|_{L_p[0,\infty)}.$$

Thus

$$\|R_2\|_{L_p(I_2)} \leq Hn^{-(k+1)} \| f \|_{L_p[0,\infty)}.$$

Now using (11) and [3], we get $\|R_1\|_{L_p(I_2)} \leq Hn^{-(k+1)} \sum_{j=0}^{2k+1} \| f^{(j)} \|_{L_p(I_2)}$

$$\leq Hn^{-(k+1)}\left(\| f^{(j)} \|_{L_p(I_2)} + \| f^{(2k+2)} \|_{L_p(I_2)}\right).$$

Combining the estimates of R_2 and R_1, we get the result

$$\|\psi_2\|_{L_p(I_2)} \leq Hn^{-(k+1)}\left(\| f^{(j)} \|_{L_p[0,\infty)} + \| f^{(2k+2)} \|_{L_p(I_2)}\right).$$

Which is the required theorem.

Theorem 2 . *Let* $f \in L_1[0,\infty)$. *If* f *has* $(2k+1)$ *derivatives on* I_1 *with* $f^{(2k)} \in AC(I_1)$ *and* $f^{(2k+1)} \in BV(I_1)$, *then for* n *sufficiently large we have*

$$\left\|S_{n,\alpha,\beta}(f,k,.)-f\right\|_{L_1(I_2)} \le Hn^{-(k+1)}\left(\left\|f^{(2k+1)}\right\|_{BV(I_2)}+\left\|f^{(2k+1)}\right\|_{L_1(I_2)}+\left\|f\right\|_{L_p[0,\infty)}\right),$$

where the constant H is independent of n and f.

Proof: By our given hypothesis on f, and for all $x \in I_2$ and for all $t \in I_1$, we have

$$f(t)=\sum_{i=0}^{2k+1}\frac{(t-x)^i}{i!}f^{(i)}(x)+\frac{1}{(2k+1)!}\int_x^t((t-u)^{2k+1}df^{(2k+1)}(u)).$$

We can write

$$f\left(\frac{nt+\alpha}{n+\beta}\right)=\sum_{i=0}^{2k+1}\frac{\left(\frac{nt+\alpha}{n+\beta}-x\right)^i}{i!}f^{(i)}(x)+\frac{1}{(2k+1)!}\int_x^t\left(\left(\frac{nt+\alpha}{n+\beta}-u\right)^{2k+1}df^{(2k+1)}(u)\right)\phi\left(\frac{nt+\alpha}{n+\beta}\right)$$

$$+F\left(\frac{nt+\alpha}{n+\beta},x\right)\left(1-\phi\left(\frac{nt+\alpha}{n+\beta}\right)\right).$$

where $\phi(t)$ being the characteristic function on I_1.

$$F\left(\frac{nt+\alpha}{n+\beta},x\right)=f\left(\frac{nt+\alpha}{n+\beta}\right)-\sum_{i=0}^{2k+1}\frac{(\frac{nt+\alpha}{n+\beta}-x)^i}{i!}f^{(i)}(x),$$

For all $t \in [0,\infty)$ and $x \in I_2$. Therefore we have

$$S_{n,\alpha,\beta}(f,k,x)-f(x)=\sum_{i=0}^{2k+1}\frac{f^{(i)}(x)}{i!}S_{n,\alpha,\beta}\left(\left(\frac{nt+\alpha}{n+\beta}-x\right)^i,k,x\right)$$

$$+\frac{1}{(2k+1)!}S_{n,\alpha,\beta}\left[\int_x^t\left(\frac{nt+\alpha}{n+\beta}-u\right)^{2k+1}df^{(2k+1)}(u)\phi\left(\frac{nt+\alpha}{n+\beta}\right),k,x\right]$$

$$+S_{n,\alpha,\beta}\left(F\left(\frac{nt+\alpha}{n+\beta},x\right)\right)\left(1-\phi\left(\frac{nt+\alpha}{n+\beta}\right),k,x\right)=:R_1+R_2+R_3.$$

Applying Lemma 1 and [3], we get

$$\left\|R_1\right\|_{L_1(I_2)} \le Hn^{-(k+1)}\left(\left\|f\right\|_{L_1(I_2)}\left\|f^{(2k+1)}\right\|_{L_1(I_2)}\right).$$

Now we have,

$$G=\left\|S_{n,\alpha,\beta}\left[\int_x^t\left(\frac{nt+\alpha}{n+\beta}-u\right)^{2k+1}df^{(2k+1)}(u)\phi\left(\frac{nt+\alpha}{n+\beta}\right),x\right]\right\|_{L_1(I_2)}$$

$$\leq \int_{a_2}^{b_2}\int_{a_1}^{b_1} W(n,x,t)\left|\left(\frac{nt+\alpha}{n+\beta}-x\right)\right|^{2k+1}\left|\int_x^t \left|df^{(2k+1)}(u)\right|\right| dtdx \,..$$

For each n there exists a nonnegative integer $r = r(n)$ such that $rn^{-1/2} < \max\,(b_1 - a_2, b_2 - a_1) \leq (r+1)\,n^{-1/2}$. Then we have

$$G \leq \sum_{l=0}^{r}\int_{a_2}^{b_2}\{\int_{x+(l)n^{-1/2}}^{x+(l+1)n^{-1/2}} \phi\left(\frac{nt+\alpha}{n+\beta}\right)W(n,x,t)\left|\left(\frac{nt+\alpha}{n+\beta}-x\right)\right|^{2k+1} \times\left[\int_x^{x+(l+1)n^{-1/2}}\phi(u)\left|df^{(2k+1)}(u)\right|\right]dt$$

$$+\int_{x-(l+1)n^{-1/2}}^{x-ln^{-1/2}}\phi\left(\frac{nt+\alpha}{n+\beta}\right)W(n,x,t)\left|\left(\frac{nt+\alpha}{n+\beta}-x\right)\right|^{2k+1} \times\left[\int_{x-(l+1)n^{-1/2}}^{x}\phi(u).\left|df^{(2k+1)}(u)\right|\right]dt\,\}\,dx\,.$$

Let $\phi_{x,c,d}(u)$ be the characteristic function of the interval $[x - cn^{-1/2}, x + dn^{-1/2}]$, where c, d are non-negative integers. Hence we get

$$G \leq \sum_{l=1}^{r}\int_{a_2}^{b_2}\{\int_{x+(l)n^{-1/2}}^{x+(l+1)n^{-1/2}} \phi\left(\frac{nt+\alpha}{n+\beta}\right)W(n,x,t)l^{-4}n^{2}\left|\left(\frac{nt+\alpha}{n+\beta}-x\right)\right|^{2k+5}$$

$$\times\left[\int_x^{x+(l+1)n^{-1/2}}\phi(u)\phi_{x,0,l+1}(u)\left|df^{(2k+1)}(u)\right|\right]dt$$

$$+\int_{x-(l+1)n^{-1/2}}^{x-ln^{-1/2}}\phi\left(\frac{nt+\alpha}{n+\beta}\right)W(n,x,t)l^{-4}n^{2}\left|\left(\frac{nt+\alpha}{n+\beta}-x\right)\right|^{2k+5}$$

$$\times\left[\int_{x-(l+1)n^{-1/2}}^{x}\phi(u)\phi_{x,l+1,0}(u)\left|df^{(2k+1)}(u)\right|\right]dt\,\}\,dx$$

$$+\int_{a_2}^{b_2}\int_{-n^{-1/2}}^{a_1+n^{-1/2}}\phi\left(\frac{nt+\alpha}{n+\beta}\right)W(n,x,t)\left|\left(\frac{nt+\alpha}{n+\beta}-x\right)\right|^{2k+1}$$

$$\times\left[\int_{l-n^{-1/2}}^{x+n^{-1/2}}\phi(u)\phi_{x,l,1}(u)\left|df^{(2k+1)}(u)\right|\right]dtdx$$

$$\leq \sum_{l=1}^{r}[l^{-4}n^{2}\int_{a_2}^{b_2}\{\int_{x+(l)n^{-1/2}}^{x+(l+1)n^{-1/2}} \phi\left(\frac{nt+\alpha}{n+\beta}\right)W(n,x,t)\left|\left(\frac{nt+\alpha}{n+\beta}-x\right)\right|^{2k+5}$$

$$\times\left(\int_{a_1}^{b_1}\phi(u)\phi_{x,0,l+1}(u)\left|df^{(2k+1)}(u)\right|\right)dt\,]$$

$$+\int_{x-(l+1)n^{-1/2}}^{x-(l)n^{-1/2}}\phi\left(\frac{nt+\alpha}{n+\beta}\right)W(n,x,t)\left|\left(\frac{nt+\alpha}{n+\beta}-x\right)\right|^{2k+5}$$

$$\times\left(\int_{a_1}^{b_1}\phi_{x,l+1,0}(u)\left|df^{(2k+1)}(u)\right|\right)dt\,\}\,dx$$

$$+\int_{a_2}^{b_2}\int_{-n^{-1/2}}^{a_1+n^{-1/2}}\phi\left(\frac{nt+\alpha}{n+\beta}\right)W(n,x,t)\left|\left(\frac{nt+\alpha}{n+\beta}-x\right)\right|^{2k+1}$$

$$\times\left[\int_{a_1}^{b_1}\phi_{x,l,1}(u)\left|df^{(2k+1)}(u)\right|dtdx\right].$$

Further using Lemma 1 and Fubini's theorem, we obtain

$$G \le Hn^{-(2k+1)/2}[\sum_{l=1}^{r} l^{-4}[\int_{a_2}^{b_2}\int_{a_1}^{b_1}\phi_{x,0,l+1}(u)\left|df^{(2k+1)}(u)\right|dx$$

$$+\int_{a_2}^{b_2}\int_{a_1}^{b_1}\phi_{x,l+1,0}(u)\left|df^{(2k+1)}(u)\right|dx+\int_{a_2}^{b_2}\int_{a_1}^{b_1}\phi_{x,l,1}(u)\left|df^{(2k+1)}(u)\right|dx]$$

$$= Hn^{-(2k+1)/2}[\sum_{l=1}^{r} l^{-4}\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\phi_{x,0,l+1}(u)\,dx\left|df^{(2k+1)}(u)\right|\right)$$

$$+\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\phi_{x,0,l+1}(u)\,dx\right)\left|df^{(2k+1)}(u)\right|)+\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\phi_{x,l,1}(u)\,dx\right)\left|df^{(2k+1)}(u)\right|$$

$$\le Hn^{-(2k+1)/2}[\sum_{l=1}^{r} l^{-4}\left[\int_{a_2}^{b_2}\int_{w-(l+1)n^{-1/2}}^{w}dx\right]\left|df^{(2k+1)}(u)\right|$$

$$+\int_{a_1}^{b_1}\left(\int_{w}^{w+(l+1)n^{-1/2}}dx\right)\left|df^{(2k+1)}(u)\right|+\int_{a_1}^{b_1}\left(\left(\int_{w-n^{-1/2}}^{w+n^{-1/2}}dx\right)\left|df^{(2k+1)}(u)\right|\right)$$

$$\le Hn^{-(k+1)}\left\|f^{(2k+1)}\right\|_{BV(I_1)}.$$

Hence, $\left\|R_2\right\|_{L_p(I_2)} \le Hn^{-(k+1)}\left\|f^{(2k+1)}\right\|_{BV(I_1)}$, where the constant H depends on k.

For $t\in[0,\infty)\setminus[a_1,b_1],\ x\in I_2$, there exist a $\delta>0$ such that $|t-x|\ge\delta$. Then

$$\left\|S_{n,\alpha,\beta}\left(F(t,x)\left(1-\phi\left(\frac{nt+\alpha}{n+\beta}\right),x\right)\right)\right\|_{L_p(I_2)} \le \int_{a_2}^{b_2}\int_0^{\infty}W(n,x,t)\left|f(t)\right|^p\left(1-\phi\left(\frac{nt+\alpha}{n+\beta}\right)\right)dtdx$$

$$+\sum_{i=0}^{2k+1}\frac{1}{i!}\int_{a_2}^{b_2}\int_0^{\infty}W(n,x,t)\left|f^{(i)}(x)\right|\left|\frac{nt+\alpha}{n+\beta}-x\right|^i\times$$

$$\times\left(1-\phi\left(\frac{nt+\alpha}{n+\beta}\right)\right)dtdx$$

$$= R_4+R_5.$$

Now for sufficiently large t, $\exists$ positive constant N_0 and H. such that $\dfrac{\left(\frac{nt+\alpha}{n+\beta}-x\right)^{2k+2}}{\left(\frac{nt+\alpha}{n+\beta}\right)^{2k+2}+1} > H$,

for all $t\ge M_0,\ x\in I_2$.

Now using Fubini's theorem

$$R_4=\left[\int_0^{N_0}\int_{a_2}^{b_2}+\int_{N_0}^{0}\int_{a_2}^{b_2}\right]W(n,x,t)\left|f\left(\frac{nt+\alpha}{n+\beta}\right)\right|\left(1-\phi\left(\frac{nt+\alpha}{n+\beta}\right)\right)dtdx = R_6+R_7.$$

Now by using Lemma 1, we have

$$R_6 = \delta^{-(2k+2)} \int_0^{N_0} \int_{a_2}^{b_2} W(n,x,t) \left| f\left(\frac{nt+\alpha}{n+\beta}\right) \right| \left(\frac{nt+\alpha}{n+\beta} - x \right)^{(2k+2)} dtdx$$

$$\leq Hn^{-(k+1)} \left[\int_0^{N_0} \left| f\left(\frac{nt+\alpha}{n+\beta}\right) \right| dt \right].$$

$$R_7 = \frac{1}{H} \int_{N_0}^{\infty} \int_{a_2}^{b_2} W(n,x,t) \frac{\left(\frac{nt+\alpha}{n+\beta} - x \right)^{2k+2}}{\left(\frac{nt+\alpha}{n+\beta} \right)^{2k+2} + 1} \left| f\left(\frac{nt+\alpha}{n+\beta}\right) \right| dtdx \leq Hn^{-(k+1)} \left[\int_{N_0}^{\infty} \left| f\left(\frac{nt+\alpha}{n+\beta}\right) \right| dt \right].$$

Now combining the estimates of R_6 and R_7, we get $R_4 \leq Hn^{-(k+1)} \|f\|_{L_1[0,\infty)}$.

By using (11) and [3], we get

$$R_5 \leq \delta^{-(2k+2)} \sum_{i=0}^{2k+1} \frac{1}{i!} \int_{a_2}^{b_2} \int_0^{\infty} W(n,x,t) \left| f^{(i)}(x) \right| \left(\frac{nt+\alpha}{n+\beta} - x \right)^{2k+i+2} dtdx$$

$$\leq Hn^{-(k+1)} \left(\sum_{i=0}^{2k+1} \left\| f^{(i)} \right\|_{L_1(I_2)} \right)$$

$$\leq Hn^{-(k+1)} \left(\|f\|_{L_1(I_2)} + \left\| f^{(2k+1)} \right\|_{L_1(I_2)} \right).$$

From above estimates of R_4 and R_5, we get

$$\left\| S_{n,\alpha,\beta} \left(F\left(\frac{nt+\alpha}{n+\beta}, x \right) \left(1 - \phi\left(\frac{nt+\alpha}{n+\beta} \right) \right), x \right) \right\|_{L_1(I_2)} \leq Hn^{-(k+1)} \left(\|f\|_{L_1[0,\infty)} + \left\| f^{(2k+1)} \right\|_{L_1(I_2)} \right).$$

Hence, we obtain

$$\|R_3\|_{L_1(I_2)} \leq Hn^{-(k+1)} \left(\|f\|_{L_1[0,\infty)} + \left\| f^{(2k+1)} \right\|_{L_1(I_2)} \right).$$

Finally combining the estimates of R_1, R_2, R_3, we obtain the required theorem.

Theorem 3. *Let* $f \in L_p[0,\infty)$, $p \geq 1$, *then for* n *sufficiently large*

$$\left\| S_{n,\alpha,\beta}(f,k,.) - f \right\|_{L_p(I_2)} \leq H \left(u_{2k+2}(f, n^{-1/2}, p, I_1) + n^{-(k+1)} \|f\|_{L_p[0,\infty)} \right),$$

where the constant H is independent of n and f.

Proof: Let $f_{\eta,2k+2}(t)$ be the Steklov mean of $(2k+2)^{th}$ order corresponding to $f(t)$ where $\eta > 0$ is sufficiently small and $f(t)$ is defined as zero outside $[0,\infty)$, then we have

$$\left\| S_{n,\alpha,\beta}(f,k,.) - f \right\|_{L_p(I_2)} \leq \left\| S_{n,\alpha,\beta}(f - f_{\eta,2k+2}, k,.) \right\|_{L_p(I_2)}$$

$$+ \left\| S_{n,\alpha,\beta}(f - f_{\eta,2k+2}, k,.) - f_{\eta,2k+2} \right\|_{L_p(I_2)} + \left\| f_{\eta,2k+2} - f \right\|_{L_p(I_2)}$$

$$=: \psi_1 + \psi_2 + \psi_3.$$

To estimate ψ_1, let $\phi(t)$ be the characteristic function of I_3, then

$$S_{n,\alpha,\beta}\left((f-f_{\eta,2k+2})\left(\frac{nt+\alpha}{n+\beta}\right),x\right)=S_{n,\alpha,\beta}\left(\phi\left(\frac{nt+\alpha}{n+\beta}\right)(f-f_{\eta,2k+2})\left(\frac{nt+\alpha}{n+\beta}\right),x\right)=:\psi_4+\psi_5.$$

Next is true for $p=1$, and $p>1$ according to Holder's inequality

$$\int_{a_2}^{b_2}|\psi_4|^p\,dt\le\int_{a_2}^{b_2}\int_{a_3}^{b_3}W(n,x,t)\left|(f-f_{\eta,2k+2})\left(\frac{nt+\alpha}{n+\beta}\right)\right|^p dxdt\ .$$

Applying Fubini's theorem, we get

$$\int_{a_2}^{b_2}|\psi_4|^p\,dt\le\int_{a_2}^{b_2}\int_{a_3}^{b_3}W(n,x,t)\left|(f-f_{\eta,2k+2})\left(\frac{nt+\alpha}{n+\beta}\right)\right|^p dxdt\ .$$

$$\le\left\|f-f_{\eta,2k+2}\right\|_{L_p(I_3)}^p.$$

Hence

$$\left\|\psi_4\right\|_{L_p(I_2)}^p\le\left\|f-f_{\eta,2k+2}\right\|_{L_p(I_3)}^p.$$

Applying Holder'inequality, (11) and Fubini's theorem , for $p\ge1$ we get the results.

$$\left\|\psi_5\right\|_{L_p(I_2)}\le Hn^{-(k+1)}\left\|f-f_{\eta,2k+2}\right\|_{L_p[0,\infty)}.$$

By using Jenson's inequality and Fubini's theorem, we obtain

$$\left\|f_{\eta,2k+2}\right\|_{L_p[0,\infty)}\le H\left\|f\right\|_{L_p[0,\infty)}.$$

Hence

$$\left\|\psi_5\right\|_{L_p[0,\infty)}\le Hn^{-(k+1)}\left\|f\right\|_{L_p[0,\infty)}.$$

Now using 3^{rd} property of Steklov mean, we get

$$\psi_1\le H\left(u_{2k+2}(f,n,p,I_1)+n^{-(k+1)}\left\|f\right\|_{L_p[0,\infty)}\right).$$

We know that.

$$\left\|f_{\eta,2k+2}^{(2k+1)}\right\|_{BV(I_3)}=\left\|f_{\eta,2k+2}^{(2k+1)}\right\|_{L_1(I_3)}.$$

According to Theorem 1, Theorem 2 and Lemma 3, we have

$$\psi_2\le Hn^{-(k+1)}\left(\left\|f_{\eta,2k+2}^{(2k+2)}\right\|_{L_p(I_3)}+\left\|f_{\eta,2k+2}\right\|_{L_p[0,\infty)}\right)$$

$$\le H\left(\eta^{-(2k+2)}u_{2k+2}(f.n,p,I_1)+n^{-(k+1)}\left\|f\right\|_{L_p[0,\infty)}\right),$$

To estimate ψ_3, we use the 3^{rd} property of Steklov mean, and obtain that s

$$\psi_3\le Hu_{2k+2}(f,n,p,I_1).$$

which is the required result and completes the proof of above theorem.

4. Conclusion

The modification of operators plays an important role in approximation theory to obtain better approximation. In this paper, we present a direct theorem for the linear combina- tion of stancu type operators, we use the technique of linear approximation method.

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