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Some Elementary Problems from the Note Books of Srinivasa Ramanujan IV

N.Ch.Pattabhi Ramacharyulu

Professor (Rtd) NIT Warangal
Warangal,A.P
pattabhi1933@yahoo.com

Problem 41: **NBSR Vol II p 386**

For any number x, $(x^4+1)^4 + 4\left(\frac{x^5-5x}{2}\right)^4 + 5(x^4-2)^4 = 3^4 + 4\left(\frac{x^5+x}{2}\right)$

Solution:

L.H.S. of * = $(x^4+1)^4 + 5(x^4-2)^4 + 4\left(\frac{x^5-5x}{2}\right)^4$

$$= [x^{16}+4x^{12}+6x^8+4x^4+1] + 5[x^{16}-8x^{12}+24x^8-32x^4+16]$$
$$+\frac{x^4}{4}[x^{16}-20x^{12}+150x^8- \qquad]$$

$$= 81 + \frac{x^4}{4}[x^{16}+4x^{12}+6x^8+\ 4x^4+1]$$

$$= 3^4 + \frac{x^4}{4}(x^4+1)^4 = 3^4 + 4\left(\frac{x^5+x}{2}\right)^4 = R.H.S$$

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Problem 42: **NBSR Vol II p 106**

$$(4x^5-5x)^4 + (4x^4+1)^4 + 5(4x^4-x)^4 = 3^4 + (4x^5+x)^4$$

Solution:

$$\text{L.H.S} = (4x^5-5x)^4 + (4x^4+1)^4 + 5(4x^4-2)^4$$
$$= x^4(4x^4-5)^4 + (4x^4+1)^4 + 5(4x^4-2)^4$$

$$= x^4(256x^{16}-1280x^{12}+2400x^8-2000x^4+625)$$
$$+ (256x^{16}+256x^{12}+96x^8+16x^4+1)$$
$$+ 5(256x^{16}-512x^{12}+384x^8-128x^4+16)$$

$$= 81 + x^4(256x^{16}+256x^{12}+96x^8+16x^4+1)$$
$$= 81 + x^4(4x^4+1)^4 = 81 + (4x^5+x)^4$$ = R.H.S. of *

Note: This result can be realized by replacing x by $(\sqrt{2}x)$ in the problem no. 1*.

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Problem 43: **NBSR Vol II p 386**

$3^4 + (2x^4 - 1)^4 + (4x^5 + x)^4 = (4x^4 + 1)^4 + (6x^4 - 3)^4 + (4x^5 - 5x)^4$ *

Solution: Comparing this result with the result 2, it can be hat it is sufficient to establish the identity.

$$5(4x^4 - 2)^4 = (6x^4 - 3)^4 - (2x^4 - 1)^4$$
(1)

$$L.H.S\ of\ (1) = 5(256x^{16} - 512x^{12} + 384x^8 - 128x^4 + 16)$$
$$= 1280x^{16} - 2560x^{12} + 1920x^8 - 640x^4 + 80$$
$$R.H.S\ of\ (1) = (1296x^{16} - 2592x^{12} + 1944x^8 - 648x^4 + 81)$$
$$- (16x^{16} - 32x^{12} + 24x^8 - 8x^4 + 1)$$
$$= 1280x^{16} - 2560x^{12} + 1920x^8 - 640x^4 + 80$$

This establishes the identity (1).

Substituting this in the result (2), we notice that

$$(4x^3 - 5x)^4 + (4x^4 + 1)^4 + (6x^4 - 3)^4 - (2x^4 - 1)^4 = 3^4 + (4x^5 + x)^4$$
$$i.e., 3^4 + (2x^4 - 1)^4 + (4x^5 + x)^4 = (4x^4 + 1)^4 + (6x^4 - 3)^4 + (4x^3 - 5x)^4$$

This establishes the identity*

Note: The results 1, 2 and 3 above can be verified by the direct application of the identity

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

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Problem 44: **NBSR Vol II p 386, B p 107**

$3^4 + (2x^4 - 1)^4 + (4x^5 + x)^4 = (4x^4 + 1)^4 + (6x^4 - 3)^4 + (4x^3 - 5x)^4$ *

Verification: For this the formula employed is $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$$(2x^4 - 1)^4 = (2x^4)^4 + 4(2x^4)^3(-1) + 6(2x^4)^2(-1)^2 + 4(2x^4)(-1)^3 + (-1)^4$$
$$= 16x^{16} - 32x^{12} + 24x^8 - 8x^4 + 1$$

And

$$(4x^5 + x)^4 = x^4(4x^4 + 1)^4$$
$$= x^4\{(4x^4)^4 + 4(4x^4)^3 + 6(4x^4)^2 + 4(4x^4) + 1\}$$
$$= x^4(256x^{16} + 256x^{12} + 96x^8 + 16x^4 + 1)$$
$$= 256x^{20} + 256x^{16} + 96x^{12} + 16x^8 + x^4$$

Then L.H.S of * $= 3^4 + (2x^4 - 1)^4 + (4x^5 + x)^4$

$$= 16x^{16} - 32x^{12} + 24x^8 - 8x^4 + 81 + 1 + 256x^{20}$$
$$+256x^{16} + 96x^{12} + 16x^8 + x^4$$
$$= 256x^{20} + 272x^{16} + 64x^{12} + 40x^8 - 7x^4 + 82$$

Further,

$$(4x^4 + 1)^4 = (4x^4)^4 + 4(4x^4)^3 + 6(4x^4)^2 + 4(4x^4) + 1$$
$$= 256x^{16} + 256x^{12} + 96x^8 + 16x^4 + 1$$

$$(6x^4-3)^4 = (6x^4)^4 + 4(6x^4)^3(-3) + 6(6x^4)^2(-3)^2 + 4(6x^4)(-3)^3 + (-3)^4$$
$$= 1296x^{16} - 2592x^{12} + 1944x^8 - 648x^4 + 81$$

And

$$(4x^5-5x)^4 = x^4(4x^4-5)^4$$
$$= x^4\{(4x^4)^4 + 4(4x^4)^3(-5) + 6(4x^4)^2(-5)^2 + 4(4x^4)(-5)^3 + (-5)^4\}$$
$$= x^4\{256x^{16} - 1280x^{12} + 2400x^8 - 2000x^4 + 625\}$$
$$= 256x^{20} - 1280x^{16} + 2400x^{12} - 2000x^8 + 625x^4$$

Then the R.H.S of * $= (4x^4+1)^4 + (6x^4-3)^4 + (4x^3-5x)^4$

$$= 256x^{16} + 256x^{12} + 96x^8 + 16x^4 + 1 + 1296x^{16} - 2592x^{12} + 1944x^8 - 648x^4 + 81 + 256x^{20} - 1280x^{16} + 2400x^{12} - 2000x^8 + 625x^4$$
$$= 256x^{20} + 272x^{16} + 64x^{12} + 40x^8 - 7x^4 + 82$$

$\therefore L.H.S = R.H.S$ This verifies the formula/identity *.

An important note:

The result (3) yields that sum of the 4th powers of three numbers of which is 3^4 as the sum of the 4th powers of three different numbers. By taking different values for x in this identity, we get identities of the form $3^4 + a^4 + b^4 = c^4 + d^4 + e^4$:

$x = 1:$ $3^4 + 1^4 + 5^4 = 5^4 + 3^4 + 1^4$ Trivial

$x = 2:$ $3^4 + 127^4 + 129^4 = 65^4 + 93^4 + 118^4$

$x = 3:$ $3^4 + 161^4 + 246^4 = 324^4 + 483^4 + 957^4$

$x = 4:$ $3^4 + 511^4 + 4100^4 = 1025^4 + 1533^4 + 4076^4$ ……… and so on.

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Problem 45: **Scribbling of S.R on p. 4 of NBSR Vol II**

If $p^3 + q^3 + r^3 = s^2$ (1)

Then $(pa^2 + mab - rb^2)^3 + (qa^2 - nab + sb^2)^3 + (ra^2 - mab - pb^2)^3$

$$= (sa^2 - nab + qb^2)^3 \quad (2)$$

$$where\ m = (s+q)\left(\frac{s-q}{r+p}\right)^{\frac{1}{2}}\ and\ \ n = (r-p)\left(\frac{r+p}{s-q}\right)^{\frac{1}{2}} \quad (3)$$

And *a* and *b* are arbitrary quantities.

Proof:

Note: The expressions in each of the brackets ( ) in (2) are homogeneous quadratics in *a* and *b*.

Consider the expression:

$$F(a,b) = (pa^2 + mab - rb^2)^3 + (qa^2 - nab + sb^2)^3 + (ra^2 - mab - pb^2)^3 - (sa^2 - nab + qb^2)^3 \quad (4)$$

$F(a,0) = (p^3 + q^3 + r^3 - s^3)a^6 = 0$ using the condition (1).

Now when $b \neq 0$, $F(a,b) = b^6 f(x)$ say (5)

In the above equation (5) $x = a/b$ (6)

And

$$f(x) = (px^2 + mx - r)^3 + (qx^2 - nx + s)^3 + (rx^2 - mx - p)^3 - (sx^2 - nx + q)^3 \quad (7)$$

We have to establish that $F(a, 0) = 0$ *i.e.* $f(x) = 0$ under the conditions (1) and (3) stated in the problem, for all values of a and * for any .

Characterization of $f(x)$:

(i) From (7), it can be noted that

$$f\left(\frac{1}{x}\right) = -\frac{f(x)}{x^6} \quad i.e.\ x^6 f\left(\frac{1}{x}\right) = -f(x)$$

From this follows that $f(1) = -f(1)$ and $f(-1) = -f(-1) \Rightarrow f(1) = 0 = f(-1)$

$\therefore (x-1)$ and $(x+1)$ are both factors of $f(x)$and so (x^2-1) is a factor of $f(x)$.

(ii) $f(0) = (-r)^3 + s^3 + (-p)^3 - q^3 = s^2 - (p^3 + q^3 + r^3) = 0$ ($\because$condition (1))

$\therefore$ The constant (term independent of *x*) in $f(x) = 0$.

(iii) Coefficient of x^6 in $f(x) = p^3 + q^3 + r^3 - s^3 = 0$

(iv) Coefficient of x^5 in $f(x) = 3[mp^2 - nq^2 - mr^2 + ns^2]$

$$= 3[(p^2 - r^2)m + (s^2 - q^2)n]$$

$$= 3\left[(p^2 - r^2)(s+q)\left\{\frac{s-q}{r+p}\right\}^{\frac{1}{2}} + (s^2 - q^2)(r-p)\left\{\frac{r+p}{s-q}\right\}^{\frac{1}{2}}\right]$$

$$= 3(r-p)(s+q)\left[-\sqrt{(r+p)(s-q)} + \sqrt{(s-q)(r+p)}\right] = 0$$

$\therefore f(x)$ is a fourth degree expression in *x* for which $x^2 - 1$ is a factor; $f(0) = 0$ *and* $f(1) = f(-1) = 0.$ *Hence* $f(x)$ *takes the form* $f(x) = Ax^2(x^2-1)$

where A is a constant that makes (8) an identity.

Let x take the value such that $x^2 = -1$ *i.e.* $x = i$

Then $f(i) = 2A$

i.e. $2A = f(i) = \{mx - (p+r)^3\}^3 - \{mx - (p+r)^3\} + \{(s-q) + nx\}^3 + \{s + q - nx\}^3$

Since $x^2 = -1$

By a straight forward simplification using the formula

$$\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2) \text{ and noting that } x^2 = -1, \text{we get}$$

$$f(i) = 2\{(s-q)^3 - (p-r)^3\} + 6\{(p+r)m^2 - (s-q)n^2\} \quad (9)$$

The first bracket of (9)$= 2\{(s-q)^3 - (p-r)^3\}$

$$= 2\{s^2 - q^2 - p^3 - s^3 - 3sq(s-q) - pr(p+r)\}$$

$$= -6\{sq(s-q) + pr(p+r)\}$$

Using the condition and the second bracket of (9) after substituting for m and n from (3) $= 6\{(s+q)^2(s-q) - (r-p)^2(r+p)\}$ $= 6\{[(s^2 + sq + q^2)(s-q) +$ *sq(s−q)−r2−rp+p2r+p−rp(r+p) =6s3−q3+sqs−q−r3+p3+rp(r+p)*

$= 6\{sq(s-q) + rp(p+q)\}$ since $s^3 = p^3 + q^3 + r^3$

Hence $2A = f(i) = 0 \Rightarrow A = 0 \Rightarrow f(x) \equiv 0.$

This establishes the identity (2) of the problem.

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Corollary: If a and b are arbitrary, then **NBSR Vol II p-266**

$$(3a^3 + 5ab - 5b^2)^3 + (4a^2 - 4ab + 6b^2)^3 + (5a^2 - 5ab + 3b^2)^3 = (6a^2 - 4ab + b^2)^3$$

Solution: Let $p = 3; q = 4; r = 5; and\ s = 6$

The choice evidently satisfies the condition $p^3 + q^3 + s^3 = 6^3$

Further $m = 5$ and $n = 4$ (from the relations (3))

Substituting these values for p, q, r, s, m and n in the identity (2) of the problem, we establish the corollary stated.

Note: (1) The identification in this corollary is given by S.R as a problem seeking solution in Journal of Indian Mathematical Society [Q.No. 44 Vol 6 (1914) p 226].

(2) This is also mentioned in Hardy G.H and Wright E.M – An Introduction to theory of Numbers p. 201 (1960).

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S.R Entry

If $a + b + c = 0$, then

(i) $2(ab + ac + bc)^2 = a^4 + b^4 + c^4$

(ii) $2(ab + ac + bc)^4 = a^4(b - c)^4 + b^4(c - a)^4 + c^4(a - b)^4$

(iii) $2(ab + ac + bc)^6 = (a^2b + b^2c + c^2a)^4 + (ab^2 + bc^2 + ca^2)^4 + (3abc)^4$

(iv) $2(ab + bc + ca)^8 = (a^3 + 2abc)^4(b - c)^4 + (b^3 + 2abc)^4(c - a)^4 + (c^3 + 2abc)^4(a - b)^4$

and so on.

Note: 1. "and so on "at the end of this entry indicates that S.R might have derived some more formulas of this type.

2. This entry provides formulas for expressing a sum of 4th powers as twice of a second, fourth, sixth and eighth powers of $ab + ac + bc\ i.e., 2(ab + ac + bc)^n; n = 2, 4, 6, 8.$

3. The sum of the bases of the fourth powers in each of the R.H.S's of the above results is equal to zero. (An interesting observation)

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Problem 46: **NBSR Vol II 385**

If $a + b + c = 0$, then $2(ab + ac + bc)^2 = a^4 + b^4 + c^4$ *

Solution: $a + b + c = 0 \Rightarrow (a + b + c)^2 = 0 \Rightarrow a^2 + b^2 + c^2 = -2(ab + ac + bc)$

$$\therefore a^4 + b^4 + c^4 = (a^2 + b^2 + c^2)^2 - 2(a^2b^2 + a^2c^2 + b^2c^2)$$
$$= 4(ab + ac + bc)^2 - 2\{(ab + ac + bc)^2 - 2(a^2bc + ab^2c + abc^2)\}$$
$$= 2(ab + ac + bc)^2 + 4abc(a + b + c)$$
$$= 2(ab + ac + bc)^2 + 4abc(0)$$
$$= 2(ab + ac + bc)^2$$

Hence the identity (*): $a^4 + b^4 + c^4 = 2(ab + ac + bc)^2$

This is a classical result that would be employed in establishing many more results.

2nd Proof:

Consider the identity (4.1):

$$(a+b+c)^4 - a^4 - b^4 - c^4$$
$$= 4(a+b+c)^2(ab+bc+ca) + 4(a+b+c)abc - 2(ab+bc+ca)^2$$

When $a+b+c=0$, this reduces to $-a^4 - b^4 - c^4 = -2(ab+bc+ca)^2$

and hence the result: $a^4 + b^4 + c^4 = 2(ab+bc+ca)^2$ (*)

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Problem47: If $a+b+c=0$, then

$$2(ab+bc+ca)^4 = a^4(b-c)^4 + b^4(c-a)^4 + c^4(a-b)^4 \quad (*)$$

Solution: Let $X = a(b-c); Y = b(c-a)$ and $Z = c(a-b)$ (1)

Then $X+Y+Z=0$ (2)

$\therefore\ X^4 + Y^4 + Z^4 = 2(XY+YZ+ZX)^2$ (3)

Now: $XY = a(b-c).b(c-a) = ab(b-c)(c-a)$

$= ab\{c(a+b) - ab - c^2\}$

$= ab\{-ab - 2c^2\}$ (Since $a+b=-c$)

$= -ab(ab+2c^2)$

Similarly $XZ = -ac(ac+2b^2)$ and $YZ = -bc(bc+2a^2)$

$\therefore\ XY+YZ+ZX = -\{ab(ab+2c^2) + ac(ac+2b^2) + bc(bc+2a^2)\}$

$= -\{(a^2b^2 + a^2c^2 + b^2c^2) + 2abc(c+b+a)\}$

$= -\{a^2b^2 + a^2c^2 + b^2c^2\}$

$= -\{(ab+ac+bc)^2 - 2abc(a+b+c)\}$

$= -(ab+ac+bc)^2$ (4)

$\therefore$ Using (1), (3) and (4) the identity (*)

$2(ab+bc+ca)^4 = a^4(b-c)^4 + b^4(c-a)^4 + c^4(a-b)^4$ can be established.

Note: This identity is a special case of Ferrarai's identity:

$$(a^2+2ac-2bc-b^2)^4 + (b^2-2ab-2ac-c^2)^4 + (c^2+2ac+2bc-a^2)^4$$
$$= 2(a^2+b^2+c^2-ab+ac+bc)^4$$

Ref: Ferrarai F: Equation indetermimie [L'Intermee Math Vol (6) 1909 p 82-83]

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Problem 48: **NBSR Vol II p 385**

If $a+b+c=0$, then

$2(ab+ac+bc)^6 = (a^2b+b^2c+c^2a)^4 + (ab^2+bc^2+ca^2)^4 + (3abc)^4$ (1)

Solution: Let

$X = a^2b+b^2c+c^2a; Y = ab^2+bc^2+ca^2$ and $Z = 3abc$ (2)

Note that $X+Y+Z = ab(a+b) + bc(b+c) + ca(c+a) + 3abc$

$= ab(-c) + bc(-a) + ca(-b) + 3abc = 0$ (3)

$\therefore\ X^4 + Y^4 + Z^4 = 2(XY+YZ+ZX)^2$ (4)

Now $XY+YZ+ZX = XY + (X+Y)Z = XY - Z^2$ (5)

And

$$XY = (a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2)$$
$$= (a^3b^3 + b^3c^3 + c^3a^3) + abc(a^3 + b^3 + c^3) + 3a^2b^2c^2$$
$$= a^3b^3 + b^3c^3 + c^3a^3 + 6a^2b^2c^2$$
$$= (ab + ac + bc)^3 + 9a^3b^3c^3$$

Since $(a^3b^3 + a^3c^3 + b^3c^3) = (ab + bc + ca)^3 + 3a^2b^2c^2$ whenever $a + b + c = 0$

$$= (ab + ac + bc)^3 + Z^2$$

$$\therefore\ X + YZ + ZX = XY - Z^2 = (ab + ac + bc)^3 \qquad (6)$$

Using the equations (2), (4) and (6), we establish the identity (1):

$$2(ab + ac + bc)^6 = (a^2b + b^2c + c^2a)^4 + (ab^2 + bc^2 + ca^2)^4 + (3abc)^4$$

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Problem 49: **NBSR Vol II p 385**

If $a + b + c = 0$, then

$$2(\mathbf{ab} + \mathbf{bc} + \mathbf{ca})^8 = (\mathbf{a}^3 + 2\mathbf{abc})^4(\mathbf{b} - \mathbf{c})^4 + (\mathbf{b}^3 + 2\mathbf{abc})^4(\mathbf{c} - \mathbf{a})^4 + (\mathbf{c}^3 + 2\mathbf{abc})^4(\mathbf{a} - \mathbf{b})^4 \qquad (*)$$

Solution:

Let $X = (a^3 + 2abc)(b - c);\ \ Y = (b^3 + 2abc)(c - a)\ \ and\ \ Z = (c^3 + 2abc)(a - b)$
(1)

Then

$$X + Y + Z = a^3(b - c) + b^3(c - a) + c^3(a - b) + 2abc\{(b - c) + (c - a) + (a - b)\}$$
$$= (a^3b - ab^3) + (b^3c - bc^3) + (c^3a - ca^3)$$
$$= ab(a + b)(a - b) + bc(b + c)(b - c) + ca(c + a)(c - a)$$
$$= -abc\{(a - b) + (b - c) + (c - a)\} = 0 \qquad (2)$$

and therefore $X^4 + Y^4 + Z^4 = 2(XY + YZ + ZX)^2$ (3)

Now $XY = ab(a^2 + 2bc)(b^2 + 2ca)(b - c)(c - a)$

$$= ab\{a^2b^2 + 2c(a^3 + b^3) + 4ab^2c^2\}\{c(a + b) - ab - c^2\}$$
$$= ab\{a^2b^2 + 2c(3abc - c^3) + 4ab^2c^2\}\{-ab - 2c^2\}$$
$$= -\{a^4b^4 + 12a^3b^3c^2 + 18a^2b^2c^4 - 4abc^6\}$$

(4)

Similar expressions for YZ and ZX can be written by cyclic symmetry using (4).

Adding these three results, we get $XY + YZ + ZX = -[(a^4b^4 + b^4c^4 + c^4a^4) +$
12a2b2c2ab+ac+bc+
18a2b2c2a2+b2+c2−4abc(a5+b5+c5)=
$[\{(ab + ac + bc)^4 + 4a^2b^2c^2(ab + ac + bc)\} + 12a^2b^2c^2(ab + ac + bc)$

$+18a^2b^2c^2\{-2(ab + ac + ca)\} -\ 4abc\{-5abc(ab + ac + bc)\}] = -[ab + ac +$
bc4 (5)

In view all the above results (1) – (5) the truth of the identity *:

$$2(ab + ac + bc)^8 = (a^3 + 2abc)^4(b - c)^4 + (b^3 + 2abc)^4(c - a)^4 + (c^3 + 2abc)^4(a - b)^4$$ Is established.

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Problem50: **Ferrai (1909) – Garardin (1911) Identity**

If $a + b + c = 0$, then

$$(a^2 + 2ac - 2bc - b^2)^4 + (b^2 - 2ab - 2ac - c^2)^4 + (c^2 + 2ab + 2bc - a^2)^4$$
$$= 2(a^2 + b^2 + c^2 - ab + ac + bc)^4 \qquad (*)$$

Solution: Let

$$X = a^2 + 2ac - 2bc - b^2 = (a + c)^2 - (b + c)^2 = b^2 - a^2 \qquad (1.1)$$
$$Y = b^2 - 2ab - 2ac - c^2 = (b - a)^2 - (a + c)^2$$
$$= (b - a)^2 - (-b)^2 = a^2 - 2ab \qquad (1.2)$$

And

$$Z = c^2 + 2ab + 2bc - a^2 = (b + c)^2 - (a - b)^2$$
$$= (-a)^2 - (a - b)^2 = 2ab - b^2 \qquad (1.3)$$

It can be noted that

$$X + Y + Z = 0 \qquad (2)$$

Further,

$$a^2 + b^2 + c^2 - ab + ac + bc = \frac{1}{2}[(a + c)^2 + (b + c) + (a - b)^2]$$
$$= \frac{1}{2}[(-b)^2 + (-a)^2 + (a - b)^2] = a^2 - ab + b^2 \qquad (3)$$

Then the L.H.S of the identity * $= X^4 + Y^4 + Z^4$

$$= 2(XY + XZ + YZ)^2 = (XY - Z^2)^2 \qquad (4)$$

And

$$XY - Z^2 = (b^2 - a^2)(a^2 - 2ab) - (2ab - b^2)^2$$
$$= (b^2a^2 - 2ab^3 - a^4 + 2a^3b) - (4a^2b^2 - 4ab^3 + b^4)$$
$$= -(a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + b^4)$$
$$= -(a^2 - ab + b^2)^2 \qquad (5)$$

From the equations (1) – (5), we notice the validity of the identity *.

Note: The identity * can be written as

$$\{(a - c)^2 - (b - c)^2\}^4 + \{(b - a)^2 - (c - a)^2\} + \{(b - c) - (b - a)\}^4$$
$$= 2\{(c - b)(a - c) + (c - b)(b - a) + (a - c)(b - a)\}^4$$

Replace in this $c - b, a - c, b - a$ by a, b, c respectively, we then get

$$(b^2 - a^2)^4 + (c^2 - b^2)^4 + (a^2 - c^2)^4 = 2(ab + ac + bc)^4$$

i.e., $$a^4(b - c)^4 + b^4(c - a)^4 + c^4(a - b)^4 = 2(ab + ac + bc)^4$$

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Problem 51: **(An S.R type problem not from NBSR)**

If $a + b + c = 0$, then

$$2(ab + ac + bc)^8 = \left\{\frac{a^3b + b^3c + c^3a}{\sqrt{3}}\right\}^4 + \left\{\frac{ab^3 + bc^3 + ca^3}{\sqrt{3}}\right\}^4 + \left\{\frac{a^4 + b^4 + c^4}{\sqrt{3}}\right\}^4$$

Solution:

Let $X = a^3b + b^3c + c^3a$; $Y = ab^3 + bc^3 + ca^3$ and $Z = a^4 + b^4 + c^4$ (1)

Then

$$X + Y + Z = ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2) + a^4 + b^4 + c^4$$
$$= ab\{a^2 + b^2 + c^2 - c^2\} + bc\{a^2 + b^2 + c^2 - a^2\} + ca\{a^2 + b^2 + c2-b2+a4+b4+c4$$
$$= (ab + bc + ca)(a^2 + b^2 + c^2) - abc(c + a + b) + a^4 + b^4 + c^4$$
$$= (ab + ac + bc)\{-2(ab + ac + bc)\} + 2(ab + ac + bc)^2$$
$$= -2(ab + ac + bc)^2 + 2(ab + ac + bc)^2$$
$$= 0 \quad (2)$$

$$\therefore \; X^4 + Y^4 + Z^4 = 2(XY + YZ + ZX)^2 \quad (3)$$

Now $XY + XZ + YZ = XY - Z^2$

$$XY = (a^3b + b^3c + c^3a)(ab^3 + bc^3 + ca^3)$$
$$= a^4b^4 + a^3b^2c^3 + a^6bc + b^4c^4 + a^3b^3c^2 + ab^6c + c^4a^4 + a^2b^3c^2 + abc^6$$
$$= \{a^4b^4 + b^4c^4 + c^4a^4\} + a^2b^2c^2(ab + ac + bc) + abc(a^5 + b^5 + c^5)$$

And

$$Z^2 = a^8 + b^8 + c^8 + 2(a^4b^4 + a^4c^4 + b^4c^4)$$
$$\therefore XY + YZ + ZX = XY - Z^2$$
$$= -(a^8 + b^8 + c^8) - (a^4b^4 + a^4c^4 + b^4c^4) + a^2b^2c^2(ab + ac + bc) \quad +abc(a^5 + b5+c5$$
$$= 8a^2b^2c^2(ab + ac + bc) - 2(ab + ac + bc)^4 - (ab + ac + bc)^4$$
$$-4a^2b^2c^2(ab + ac + bc) + a^2b^2c^2(ab + ac + bc) + abc\{-5abc(ab + ac + bc)\}$$
$$= -3(ab + ac + bc)^4$$

Substituting these values in (3), we have

$$(a^3b + b^3c + c^3a)^4 + (ab^3 + bc^3 + ca^3)^4 + (a^4 + b^4 + c^4)^4$$
$$= -2\{-3(ab + ac + bc)^4\}^2$$
$$= 18(ab + ac + bc)^8$$

Dividing by 9 and interchanging the L.H.S and R.H.S, we write

$$2(ab + bc + ca)^8 = \frac{1}{9}\{(a^3b + b^3c + c^3a)^4 + (ab^3 + bc^3 + ca^3)^4 + (a^4 + b^4 + c^4)^4\}$$

$$= \left\{\frac{\mathbf{a^3b + b^3c + c^3a}}{\sqrt{3}}\right\}^4 + \left\{\frac{\mathbf{ab^3 + bc^3 + ca^3}}{\sqrt{3}}\right\}^4 + \left\{\frac{\mathbf{a^4 + b^4 + c^4}}{\sqrt{3}}\right\}^4$$

This establishes the result which is an S.R type result (not from NBSR).

Note: Many more identities of this type can be conceived.

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Scribbling of S.R p. 338 Vol.II NBSR

I. $\dfrac{x^5 - a}{x^2 - y} = \dfrac{y^5 - b}{y^2 - x} 5(xy - 1)$

II. $\frac{x^7-a}{(x^2-y)^2+x} = \frac{y^7-b}{(y^2-x)^2+y} = 7(xy-1)$

Problem: Solve the equation pair: $\frac{x^5-a}{x^2-y} = \frac{y^5-b}{y^2-x} 5(xy-1)$ (1)

for x and y, where a and b are arbitrary.

Solution: Let α, β, γ be three quantities satisfying the restriction

$$\alpha.\beta.\gamma = 1 \tag{2}$$

but otherwise arbitrary. Further, we take

$$x = \alpha + \beta + \gamma \; and \; y = \alpha\beta + \beta\gamma + \gamma\alpha \tag{3}$$

From (1) $x^5 - a = 5(x^2 - y)(xy - 1)$ (4.1)

and $y^5 - b = 5(y^2 - x)(xy - 1)$ (4.2)

Substituting x and y from (3) in the above two equations,

$$(\alpha+\beta+\gamma)^5 - a = 5\{(\alpha+\beta+\gamma)^2 - (\alpha\beta+\beta\gamma+\gamma\alpha)\}$$
$$\{(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha) - 1\} \tag{5.1}$$

and

$$(\alpha\beta+\beta\gamma+\gamma\alpha)^5 - b = 5\{(\alpha\beta+\beta\gamma+\gamma\alpha)^2 - (\alpha+\beta+\gamma)\}$$
$$\{(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha) - 1\} \tag{5.2}$$

Let us recall the identities for $(\alpha+\beta+\gamma)^5$ and $(\alpha\beta+\beta\gamma+\gamma\alpha)^5$, coupled with the restriction (2): $\alpha\beta\gamma = 1$: given in Appendix No. 5;

The equations (1) and (2)of Appendix (5) are

$$(\alpha+\beta+\gamma)^5 = \alpha^5+\beta^5+\gamma^5 + 5\{(\alpha+\beta+\gamma)^2 - (\alpha\beta+\beta\gamma+\gamma\alpha)\}$$
$$\{(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha) - 1\} \tag{6.1}$$

and

$$(\alpha\beta+\beta\gamma+\gamma\alpha)^5$$
$$= (\alpha\beta)^5 + (\beta\gamma)^5 + (\gamma\alpha)^5 + 5\{(\alpha\beta+\beta\gamma+\gamma\alpha)^2 - (\alpha+\beta+\gamma)\}$$
$$\{(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha) - 1\} \tag{6.2}$$

The equation (5.1) and (5.2) now reduce to

$$\alpha^5+\beta^5+\gamma^5 = a \tag{7.1}$$

$$(\alpha\beta)^5 + (\beta\gamma)^5 + (\gamma\alpha)^5 = b \tag{7.2}$$

together with the restriction (2) which can be written as

$$\alpha^5\beta^5\gamma^5 = 1 \tag{7.3}$$

The three equations (7.1) – (7.3) suggest that α^5, β^5 and γ^5 are the roots of the cubic equation:

$$t^3 - at^2 + bt - 1 = 0 \tag{8}$$

Denoting the roots of this cubic by $r_1(=\alpha^5)$; $r_2(=\beta^5)$ and $r_{3\gamma}(=\gamma^5)$ then their fifth roots are

$$\alpha = r_1^{1/5} \; ; \; \beta = r_2^{1/5} \; and \; \gamma = r_3^{1/5} \tag{9}$$

It can also be recalled that

$$t = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\ ,\ t^5 = 1 \tag{10}$$

is the primitive complex fifth root of unity. Let this quantity be denoted by ω. Then $\omega^5 = 1$.

The equations (7.1) – (7.3) remain unchanged if α, β and γ are replaced by $\alpha\omega^m$; $\beta\omega^n$ and $\gamma\omega^p$ respectively whenever m + n + p = a multiple of 5 for $(m, n, p) = (0, 1, 2, 3, 4)$.

∴ The possible values of x are now $\alpha+\beta+\gamma$; $\alpha+\beta\omega+\gamma\omega^4$; $\alpha\omega+\beta+\gamma\omega^4$; $\alpha\omega+\beta\omega^4+\gamma$; $\alpha\omega+\beta\omega+\gamma\omega^3$; $\alpha\omega+\beta\omega^3+\gamma\omega$; $\alpha\omega^3+\beta\omega+\gamma\omega$ (11)

In fact there are 25 solutions for x and correspondingly 25 values for y and the corresponding values of y are $\alpha\beta+\beta\gamma+\gamma\alpha$; $\alpha\beta\omega+\beta\gamma+\gamma\alpha\omega^4$; $\alpha\beta\omega+\beta\gamma\omega^4+\gamma\alpha$; $\alpha\beta+\beta\gamma\omega^4+\gamma\alpha\omega$; $\alpha\beta\omega^2+\beta\gamma\omega^4+\gamma\alpha\omega^4$; $\alpha\beta\omega^4+\beta\gamma\omega^4+\gamma\alpha\omega^2$; $\alpha\beta\omega^4+\beta\gamma\omega^2+\gamma\alpha\omega^4$

The complete set of solutions (x, y) of the problem are presented here under.

Table showing the roots (x, y) of the given pair of equations (1).

x	y
$\alpha+\beta+\gamma$	$\alpha\beta+\beta\gamma+\gamma\alpha=\alpha\beta+\beta\gamma+\gamma\alpha$
$\alpha+\beta\omega+\gamma\omega^4$	$\alpha\beta\omega+\beta\gamma\omega^5+\gamma\alpha\omega^4=\alpha\beta\omega+\beta\gamma+\gamma\alpha\omega^4$
$\alpha\omega+\beta\omega^2+\gamma\omega^3$	$\alpha\beta\omega^2+\beta\gamma\omega^5+\gamma\alpha\omega^3=\alpha\beta\omega^2+\beta\gamma+\gamma\alpha\omega^3$
$\alpha+\beta\omega^3+\gamma\omega^2$	$\alpha\beta\omega^3+\beta\gamma\omega^5+\gamma\alpha\omega^2=\alpha\beta\omega^3+\beta\gamma+\gamma\alpha\omega^2$
$\alpha+\beta\omega^4+\gamma\omega$	$\alpha\beta\omega^4+\beta\gamma\omega^5+\gamma\alpha\omega=\alpha\beta\omega^4+\beta\gamma+\gamma\alpha\omega$
$\alpha\omega+\beta+\gamma\omega^4$	$\alpha\beta\omega+\beta\gamma\omega^4+\gamma\alpha\omega^5=\alpha\beta\omega+\beta\gamma\omega^4+\gamma\alpha$
$\alpha\omega+\beta\omega+\gamma\omega^3$	$\alpha\beta\omega^2+\beta\gamma\omega^4+\gamma\alpha\omega^4=\alpha\beta\omega^2+\beta\gamma\omega^4+\gamma\alpha\omega^4$
$\alpha\omega+\beta\omega^2+\gamma\omega^2$	$\alpha\beta\omega^3+\beta\gamma\omega^4+\gamma\alpha\omega^3=\alpha\beta\omega^3+\beta\gamma\omega^4+\gamma\alpha\omega^3$
$\alpha\omega+\beta\omega^3+\gamma\omega$	$\alpha\beta\omega^4+\beta\gamma\omega^4+\gamma\alpha\omega^2=\alpha\beta\omega^4+\beta\gamma\omega^4+\gamma\alpha\omega^2$
$\alpha\omega+\beta\omega^4+\gamma$	$\alpha\beta\omega^5+\beta\gamma\omega^4+\gamma\alpha\omega=\alpha\beta+\beta\gamma\omega^4+\gamma\alpha\omega$
$\alpha\omega^2+\beta+\gamma\omega^3$	$\alpha\beta\omega^2+\beta\gamma\omega^3+\gamma\alpha\omega^5=\alpha\beta\omega^2+\beta\gamma\omega^3+\gamma\alpha$
$\alpha\omega^2+\beta\omega+\gamma\omega^2$	$\alpha\beta\omega^3+\beta\gamma\omega^3+\gamma\alpha\omega^4=\alpha\beta\omega^3+\beta\gamma\omega^3+\gamma\alpha\omega^4$
$\alpha\omega^2+\beta\omega^2+\gamma\omega$	$\alpha\beta\omega^4+\beta\gamma\omega^3+\gamma\alpha\omega^3=\alpha\beta\omega^4+\beta\gamma\omega^3+\gamma\alpha\omega^3$
$\alpha\omega^2+\beta\omega^3+\gamma$	$\alpha\beta\omega^5+\beta\gamma\omega^3+\gamma\alpha\omega^2=\alpha\beta+\beta\gamma\omega^3+\gamma\alpha\omega^2$
$\alpha\omega^2+\beta\omega^4+\gamma\omega^4$	$\alpha\beta\omega^6+\beta\gamma\omega^8+\gamma\alpha\omega^6=\alpha\beta\omega+\beta\gamma\omega^3+\gamma\alpha\omega$
$\alpha\omega^3+\beta+\gamma\omega^2$	$\alpha\beta\omega^3+\beta\gamma\omega^2+\gamma\alpha\omega^5 = \alpha\beta\omega^3+\beta\gamma\omega^2+\gamma\alpha$

$\alpha\omega^3 + \beta\omega + \gamma\omega$	$\alpha\beta\omega^4 + \beta\gamma\omega^2 + \gamma\alpha\omega^4$=$\alpha\beta\omega^4 + \beta\gamma\omega^2 + \gamma\alpha\omega^4$
$\alpha\omega^3 + \beta\omega^2 + \gamma$	$\alpha\beta\omega^5 + \beta\gamma\omega^2 + \gamma\alpha\omega^3$=$\alpha\beta + \beta\gamma\omega^2 + \gamma\alpha\omega^3$
$\alpha\omega^3 + \beta\omega^3 + \gamma\omega^4$	$\alpha\beta\omega^6 + \beta\gamma\omega^7 + \gamma\alpha\omega^7$=$\alpha\beta\omega + \beta\gamma\omega^2 + \gamma\alpha\omega^2$
$\alpha\omega^3 + \beta\omega^4 + \gamma\omega^3$	$\alpha\beta\omega^7 + \beta\gamma\omega^7 + \gamma\alpha\omega^6$=$\alpha\beta\omega^2 + \beta\gamma\omega^2 + \gamma\alpha\omega$
$\alpha\omega^4 + \beta + \gamma\omega$	$\alpha\beta\omega^4 + \beta\gamma\omega + \gamma\alpha\omega^5$=$\alpha\beta\omega^4 + \beta\gamma\omega + \gamma\alpha$
$\alpha\omega^4 + \beta\omega + \gamma$	$\alpha\beta\omega^5 + \beta\gamma\omega + \gamma\alpha\omega^4$=$\alpha\beta + \beta\gamma\omega + \gamma\alpha\omega^4$
$\alpha\omega^4 + \beta\omega^2 + \gamma\omega^4$	$\alpha\beta\omega^6 + \beta\gamma\omega^6 + \gamma\alpha\omega^8$=$\alpha\beta\omega + \beta\gamma\omega + \gamma\alpha\omega^3$
$\alpha\omega^4 + \beta\omega^4 + \gamma\omega^2$	$\alpha\beta\omega^8 + \beta\gamma\omega^6 + \gamma\alpha\omega^6$=$\alpha\beta\omega^3 + \beta\gamma\omega + \gamma\alpha\omega$
$\alpha\omega^4 + \beta\omega^3 + \gamma\omega^3$	$\alpha\beta\omega^7 + \beta\gamma\omega^6 + \gamma\alpha\omega^7$=$\alpha\beta\omega^2 + \beta\gamma\omega + \gamma\alpha\omega^2$

The problem now boils down to the problem of finding α, β, γ the roots of a cubic (polynomial) equation (8). The method of solving the cubic is presented in the Appendix.

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Problem 52: **NBSR Vol II p. 338**

Solve the equation pair:

$$\frac{x^7 - a}{(x^2 - y)^2 + x} = \frac{y^7 - b}{(y^2 - x)^2 + y} = 7(xy - 1) \qquad (1)$$

for x and y where a and b are arbitrary.

Solution: Let the solution be set in the form $x = \alpha + \beta + \gamma \; and \; y = \alpha\beta + \beta\gamma + \gamma\alpha$ (2)

with the restriction

$$\alpha\beta\gamma = 1 \qquad (3)$$

From (1) $x^7 - a = 7(xy - 1)\{(x^2 - y)^2 + x\}$ (4.1)

and $y^7 - b = 7(xy - 1)\{(y^2 - x)^2 + y\}$ (4.2)

Substituting the assumed solution (2) in the above two equations (4.1 , 4.2) yields

$$(\alpha + \beta + \gamma)^7 - a = 7\{(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) - 1\}$$
$$\{[(\alpha + \beta + \gamma)^2 - (\alpha\beta + \beta\gamma + \gamma\alpha)]^2 + (\alpha + \beta + \gamma)\} \qquad (5.1)$$

and $(\alpha\beta + \beta\gamma + \gamma\alpha)^7 - b = 7\{(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) - 1\}\{[(\alpha\beta + \beta\gamma + \gamma\alpha)^2 -$
$\alpha+\beta+\gamma 2+\alpha\beta+\beta\gamma+\gamma\alpha$(5.2)

Comparing these two equations (5.1) and (6.12) with the equations given Identity I.7 with the note 3 ($\alpha\beta\gamma = 1$) of the appendix

We see that

$$(\alpha)^7 + (\beta)^7 + (\gamma)^7 = a \qquad (6.1)$$

and $(\alpha\beta)^7 + (\beta\gamma)^7 + (\gamma\alpha)^7 = b$ (6.2)

together with $\alpha^7\beta^7\gamma^7 = 1$ (6.3)

∴ $\alpha^7, \beta^7, \gamma^7$ are the roots of the cubic

$$t^3 - at^2 + bt - 1 = 0 \qquad (7)$$

Let $r_1(=\alpha^7), r_2(=\beta^7)$ and $r_3(=\gamma^7)$ be the three roots of the cubic (7), then

$$\alpha = r_1^{1/7}, \beta = r_2^{1/7} \text{ and } \gamma = r_3^{1/7} \quad (8)$$

It can also be recalled that

$$s = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\ ;\ \ s^7 = 1 \quad (9)$$

is the primitive seventh root of unity.

The three equations in (7) remain unchanged when α, β, γ are replaced by $\alpha s^m, \beta s^n\ and\ \gamma s^p$ where $m + n + p$ is a multiple of 7 when $(m, n, p) = (0, 1, 2, 3, 4, 5, 6)$

m, n and p take one or other seven values 0, 1, 2, 3, 4, 5, 6.

∴ The possible values of x are now $\alpha + \beta + \gamma\ ;\ \alpha + s\beta + s^6\gamma\ ;\ s\alpha + s^6\beta + \gamma\ ;\ s\alpha + \beta + s^6\gamma\ ;$

$$s\alpha + s\beta + s^5\gamma; s\alpha + s^5\beta + s\gamma; s^5\alpha + s\beta + s\gamma;\ s\alpha + s^2\beta + s^4\gamma \text{ and } s\alpha + s^4\beta + s^2\gamma$$

and correspondingly the values of y are

$$\alpha\beta + \beta\gamma + \gamma\alpha;\ \alpha\beta s + \beta\gamma + \gamma\alpha s^6;\ \ \alpha\beta + \beta\gamma s^6 + \gamma\alpha s;\ \ \alpha\beta s + \beta\gamma s^6 + \gamma\alpha;$$

$$\alpha\beta s^2 + \beta\gamma s^6 + \gamma\alpha s^6; \alpha\beta s^6 + \beta\gamma s^6 + \gamma\alpha s^2; \alpha\beta s^6 + \beta\gamma s^2 + \gamma\alpha s^6 \text{and } \alpha\beta s^5 + \beta\gamma s^6 + \gamma\alpha s^3$$

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Problem 53: Solve the equations for $x, y\ and\ z$:

$$x + ay + a^2z + a^3 = 0$$
$$x + by + b^2z + b^3 = 0$$
$$x + cy + c^2z + c^3 = 0 \quad (*)$$

Solution: The three given equations can be written as

$$a^3 + a^2z + ay + x = 0$$
$$b^3 + b^2z + by + x = 0$$
$$c^3 + c^2z + cy + x = 0 \quad (1)$$

An inspector of these equations suggest that the roots of the equation

$$t^3 + zt^2 + yt + x = 0 \quad (2)$$

are $a, band\ c$.

∴ The sum of the roots of the equation (2)= $a + b + c = -z$

Sum of the roots taken two at a time $= ab + bc + ca = +y$ and product of the roots: $abc = -x$.

$$\therefore\ \ x = -abc\ ;\ \ y = ab + bc + ca\ \ \ and\ \ z = -(a + b + c)$$

is the solution of the given set of equations *.

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Problem 54: **NBSR Vol II p 325**

If α, β, γ be the roots of the equation

$$x^3 - ax^2 + bx - 1 - 0 \quad (1)$$

then $$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = \sqrt[3]{a + 6 + 3t} \quad (2.1)$$

and $$\sqrt[3]{\alpha\beta} + \sqrt[3]{\beta\gamma} + \sqrt[3]{\gamma\alpha} = \sqrt[3]{b + 6 + 3t} \quad (2.2)$$

where $$t^3 - 3t(a + b + 3) - \left(ab + 6.\overline{a + b} + 9\right) = 0 \quad (3)$$

Solution: α, β, γ are the roots of the cubic equation (1).

Then $\alpha+\beta+\gamma=a$; $\alpha\beta+\beta\,\gamma+\gamma\alpha=b$ *and* $\alpha\beta\gamma=1$ (4)

The results (2.1) –(2.2) follow if we can find the cubic equation whose roots are $\alpha^{1/3}, \beta^{1/3}$ *and* $\gamma^{1/3}$.

Let this cubic be

$$z^3-pz^2+qz-1=0 \quad (5)$$

The product of the three roots of the equation (5) = $\alpha^{\frac{1}{3}}\beta^{\frac{1}{3}}\gamma^{\frac{1}{3}}=(\alpha\beta\gamma)^{\frac{1}{3}}=1$ because of the last equation of (4). Hence the last term of the above cubic (5) is taken as (-1).

Then

$$\alpha^{1/3}+\beta^{1/3}+\gamma^{1/3}=p \ \text{ and } \ (\alpha\beta)^{1/3}+(\beta\gamma)^{1/3}+(\gamma\alpha)^{1/3}=q \quad (6)$$

Determination of p *and* q establishes the required result.

$$z^3-1=pz^2-qz \quad (7)$$

$$\therefore\ (z^3-1)^3=(pz^2-qz)^3$$

$$=p^3z^6-q^3z^3-3pqz^3(pz^2-qz) \quad (8)$$

which on using (7) reduces to

$$(z^3-1)^3=p^3z^6-q^3z^3-3pqz^3(z^3-1) \quad (9)$$

Assuming $u=z^3$

The equation (9) after rearranging terms, can be written as

$$u^3-(+3+p^3-3pq)u^2+(3+q^3-3pq)u-1=0 \quad (10)$$

The roots of this cubic are α,β,γ. Hence on comparing the equations (1) and (2), we notice that

$$3+p^3-3pq=a \ \text{ and } \ 3+q^3-3pq=b \quad (11)$$

Let us introduce a number *t* defined by

$$p^3=a+6+3t \quad (12)$$

Then from the first of the equations (6)

$$\alpha^{1/3}+\beta^{1/3}+\gamma^{1/3}=p=\sqrt[3]{a+6+3t} \qquad \text{n} \quad (13)$$

Also $q^3=b-3+3pq$ from the second equation of (11)

$=b-3+(3+p^3-a)$ from the first equation of (11)

$=b-a+p^3$ (substitute p^3 from (12))

$=b-a+a+6+3t$

$=b+6+3t$ (14)

Hence from (6); (12) and (14), we have

$$\alpha^{1/3}+\beta^{1/3}+\gamma^{1/3}=p=\sqrt[3]{a+6+3t}$$

And

$$(\alpha\beta)^{1/3}+(\beta\gamma)^{1/3}+(\gamma\alpha)^{1/3}=q=\sqrt[3]{b+6+3t}$$

The two relations establish the results (2.1) and (2.2). Here *t* is a parameter.

From the first of (11) and (12)

$$a+3pq-3=p^3=a+6+3t$$

$$\Rightarrow\ pq=t+3 \quad (15)$$

$\therefore \quad (t+3)^3 = (pq)^3 = p^3q^3 = (a+6+3t)(b+6+3t)$ (16)

(equations (12) and (14) are employed here)

$L.H.S\ of\ (16) = t^3 + 9t^2 + 27t + 27$

$R.H.S\ of\ (16) = 9t^2 + 3t(a+b+12) + (a+6)(b+6)$

$\therefore$ equation (16) reduces to

$$t^3 - 3t(a+b+3) + \left(ab + 6\overline{a+b} + 9\right) = 0$$

which is the required relation: (3)

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------------------ (To be continued ……)

AUTHOR'S BIOGRAPHY

Prof. N. Ch. Pattabhi Ramacharyulu: He is a retired professor in Department of Mathematics & Humanities, National Institute of Technology, Warangal. He is a stalwart of Mathematics. His yeoman services as a lecturer, professor, Professor Emeritus and Deputy Director enriched the knowledge of thousands of students. He has nearly 49 Ph.D's to his credit. His papers more than 200 were published in various esteemed reputable International Journals. He is a Member of Various Professional Bodies. He published four books on Mathematics. He received so many prestigious awards and rewards