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On Generalized (α,β)*-Derivations in *-rings

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Abstract: *In this paper, it is proved that a 2-orsion free semi-prime*-ring(semi simple *-ring) admits a generalized (α,β)*-derivation F with an associated nonzero (α,β)*-derivation d, then F maps from R into Z(R).Using these it is searched for a prime*-ring which results either F= 0 or R is commutative.*

1. Introduction

Over the last few decades, Several authors have investigated the relationship between the Commutativity of a ring R and the existence of certain specific derivations of R. (Cf.,[1],[2],[6],[9] where further references can be looked).The first result in this direction is due to posner[11] who proved that if a prime ring R admits a non-zero derivation d Such that $[d(x),x]\in Z(R)$. $\forall x\in R$, then R is commutative. An analogous result for centralizing automorphisms on prime rings was obtained by Mayne [10].A number of authors have extended these theorems of Posner and Mayne.They have showed that derivations, auto orphisms, and some related mappings cannot be centralized on certain subset of non-commutative prime and some other rings. For these results refer the reader ([2],[3],[9]) where the further references can be found. In [4] the description of all centralizing additive mappings of a prime ring R of characteristic not equal to 2 was given. See also [3] where similar results for some other rings are presented. In the year 1990, Bresar and Vukman [6] established that a prime ring must be commutative if it admits a non-zero left derivation. Further,Vukman[14] extended the above mentioned result for semi-prime rings admits a Jordan left derivation ϕ then ϕ is a derivation which maps R into Z(R).In this section our objective is to explore similar types of problems in the setting of *-rings with generalized (αβ)*-derivation.

Throughout the discussion, R will denote an associative ring with center $Z(R)$. For any $x,y\in R$, the symbol $[x,y]$ will denote the commutator $xy-yx$. We shall make extensive use of the following basic commutator identities without any specific mention:$[xy,z] = x[y,z] + [x,z]y$ and $[x,yz] = y[x,z] + [x,y]z$ for all $x,y,z\in R$. A ring R is prime if for $x,y\in R$, $xRy = \{0\}$ implies either $x = 0$ or $y = 0$, and R is semiprime if $xRx = \{0\}$ implies $x = 0$. A ring is said to be 2-torsion free if 2x = 0 then x= 0.A semi-prime *-ring is defined as $xa^*x = 0 \Rightarrow x = 0$.

An additive mapping $d : R\to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x,y\in R$. An additive mapping $x\to x^*$ of R into itself is called an involution if the following conditions are satisfied: (i) $(xy)^* = y^*x^*$, and (ii) $(x^*)^* = x$ for all $x,y\in R$. A ring equipped with an involution is called a $*$-ring or Ring with involution. Let R be a $*$-ring. An additive mapping $d: R\to R$ is said to be a $*$-derivation if $d(xy) = d(x)y^* + xd(y)$ holds for all $x,y\in R$. An additive mapping $d : R\to R$ *is said to be reverse derivation if* $d(xy)=d(y)x+y\,d(x)$ *holds for all* $x,y\in R$. *An* additive mapping $d: R\to R$ is called a reverse$*$-derivation if $d(xy) = d(y)x^* + yd(x)$ holds for all $x,y\in R$. An additive mapping d: $R\to R$ said to be $(\alpha, \beta)^*$- derivation if $d(xy)=d(x)\alpha(y^*) + \beta(x)d(y)$ holds for all $x,y\in R$. An additive mapping d: $R\to R$ is said to be reverse $(\alpha, \beta)^*$ -derivation if $d(xy) =d(y)\alpha(x^*) +\beta(y)d(x)$ holds for all $x,y\in R$. An additive mapping $F : R\to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F : R \to R$ is called a generalized *-derivation if there exists a *-derivation $d : R \to R$ such that $F(xy) = F(x)y^{*} + xd(y)$ holds for all $x, y \in R$. Let α,β be automorphisms of R. An additive mapping F : R → R is called a generalized (α,β)*-derivation with associated (α, β) *-derivation d if F (xy) =F (x) α (*) + β(x)d(y) . F is called a generalized reverse (α, β)*-derivation with associated reverse $(\alpha, \beta)^{*}$ -derivation d if F (xy) =F (y)α(x*) + β(y)d(x) holds for all $x, y \in R$. An additive mapping F : R → R is called right(resp left) α*-Centralizer if F(xy)=F(x) α(y*) (resp F(xy) = α(y*) F(x). In[5] Brešar and vukman proved that if a prime ∗-ring R admits a ∗-derivation(resp. reverse ∗-derivation) d, then either $d = 0$ or R is commutative. Further, the author Shakir Ali [13] together with Ashraf [1] extended the above mentioned result for semi prime ∗-rings. During the last few decades many authors have studied derivations in the context of prime and semi-prime rings with involution (viz., [1], [5], [7], [8], [9], and [12]).

The aim of the present paper is to establish some results involving generalized (α,β) *-derivations and generalized reverse(α,β)*-derivations. The obtained results generalizes the result given by Brešar and Vukman [5] to a large class of ∗-rings.

Next we prove the result on 2-torsion free semi-prime *-ring.

2. Main Results

Theorem2.1: Let R be a 2-torsion free semi-prime*- ring. if R admits a generalized (α,β)* -derivation F with an associated non-zero $(\alpha,\beta)^{*}$ -derivation d, then F maps from R to Z(R).

Proof: Let F be a generalized $(\alpha,\beta)^{*}$- derivation with an associated non-zero $(\alpha,\beta)^{*}$- derivation, then we have

$F(xy) = F(x)\alpha(y^*)+\beta(x)d(y)\ \forall x,y \in R$ (1)

Replacing y by yz in (1) we get

$F(xyz) = F(x)\alpha((yz)^*)+\beta(x) d(yz).$

Since d is (α,β)* -derivation then

$F(xyz)= F(x) \alpha(z^*y^*)+\beta(x) (d(y)\alpha(z^*)+\beta(y)d(z))= F(x)\alpha(z^*)\alpha(y^*)+\beta(x)d(y) \alpha(z^*)+\beta(x)\beta(y)d(z)$ (2)

On the other hand

$F(xyz)=F(xy(z))$

$= F(xy) \alpha(z^*)+ \beta(xy) d(z).$

$= F(x) \alpha(y^*)\alpha(z^*)+ \beta(x) d(y) \alpha(z^*)+ \beta(x \beta(y)d(z).$

Comparing (1) and (2) we get

$F(x) [\alpha(z^*),\alpha(y^*)]=0.$ (3)

Replacing z* by z, y* by y in (3) we get

$F(x)[\alpha(z),\alpha(y)]=0.$ (4)

Replacing z by zF(x) in (4) we get

$F(x)\alpha(z)[\alpha(F(x),\alpha(y)] +F(x) [\alpha(z),\alpha(y)] \alpha(F(x) = 0.$

Using (4) we have

$F(x)\alpha(z)[\alpha(F(x),\alpha(y)]=0.$ (5)

Left multiplication of (5) by α(yF(x)) we get

$\alpha(yF(x))\ F(x)\alpha(z) [\alpha(F(x) ,\alpha(y)] = 0.$

$\alpha(y)\alpha(F(x)F(x)\alpha(z)[\alpha(F(x),\alpha(y)] = 0.$ (6)

Left multiplication of (5) by $\alpha(F(x)y)$ we get

$\alpha(F(x)y)\ F(x)\ \alpha(z)\ [\alpha(F(x) ,\alpha(y)] = 0.$

$=\alpha(F(x)\alpha(y)\ F(x)\ \alpha(z)\ [\alpha(F(x),\alpha(y)] = 0$ (7)

Comparing (6) and (7) we get

$[\alpha(F(x),\alpha(y)]\ F(x)\ \alpha(z)\ [\alpha(F(x),\alpha(y)] = 0.$

I.e) $[\alpha(F(x),\alpha(y)]\ R\ [\alpha(F(x),\alpha(y)] = 0.\forall x,y\in R.$

Semi-Primeness of R forces the above equation to

$[\alpha(F(x),\alpha(y)] = 0.\forall x,y\in R.$

$\alpha(F(x)\ \alpha(y) - \alpha(y)\ \alpha(F(x) = 0$.

$\alpha(F(x)\ y) - \alpha(yF(x)) = 0.$

$\alpha(F(x)\ y - yF(x)) = 0.$

$\alpha\ [F(x),y] = 0.\ \forall x,y\in R$

Since $\alpha\neq 0$ is an automorphism of R we get

$[F(x),y] = 0.\ \forall x,y\in R.$

Hence F is mapping from R into Z(R). <>

Next theorem deals with a semi-prime*-ring R admits an additive mapping G from R to itself satisfying $G(xy) = G(x)\ \alpha(y^*).\forall x,y\in R$, then G maps from R to center of R.

<u>Theorem2.2:</u> Let R be a semi-prime*-ring. If $G:R\to R$ is an additive mapping such that

$G(xy)= G(x)\alpha(y^*).\forall x,y\in R$, then G maps from R to Z(R).

<u>Proof:</u> By assumption we have $G(xy)= G(x)\alpha(y^*).\forall x,y\in R.$

Now compute G(xzy) in two different ways. On the one hand

$G(xzy) = G(x(zy)) = G(x)\ \alpha((zy)^*).$

$= G(x)\ \alpha(y^*z^*) = G(x)\ \alpha(y^*)\ \alpha(z^*)$ (8)

On the other hand

$G(xzy)=G((xz)y)=G(xz)\alpha(y^*)$
$=G(x)\ \alpha(z^*)\ \alpha(y^*)$ (9)

Comparing (8) and (9) we get

$G(x)[\alpha(z^*),\alpha(y^*)]= 0.$ (10)

Replacing z^* by z, y^* by y in (10) we get

$G(x)[\alpha(z),\alpha(y)]= 0.$ (11)

Replacing z by zG(x) in (11) we get

$G(x)\ [\alpha(zG(x),\alpha(y)] = 0.$

$G(x)\ [\alpha(z)\alpha(G(x),\alpha(y)] = 0.$

$G(x)\ [\alpha(z),\alpha(y)]\ \alpha G(x)+ G(x)\ \alpha(z)\ [\alpha(G(x),\alpha(y)] = 0.$

Using (11) we obtain $G(x)\alpha(z)[\alpha(G(x),\alpha(y)] = 0.$ (12)

Left multiplication of (12) by $\alpha(yG(x))$ we get

$\alpha(yG(x))\ G(x)\alpha(z)\ [\alpha(G(x), \alpha(y)] = 0.$

$\alpha(y)\alpha(G(x)\ G(x)\ \alpha(z)[\alpha(G(x),\alpha(y)] = 0$. (13)

Left multiplication of (12) by $\alpha(G(x)y)$ we get

$\alpha(G(x)\, y)\, G(x)\, \alpha(z)\, [\, \alpha(G(x),\alpha(y)\,] = 0.$

$\alpha(G(x))\alpha(y)G(x)\alpha(z)[\, \alpha(G(x),\alpha(y)\,] = 0$ (14)

Comparing (13) and (14) we get

$[\alpha(G(x),\alpha(y)]\, R\, [\, \alpha(G(x),\alpha(y)\,] = 0.\forall x,y\in R.$

Semi-Primeness of R forces the above equation to

$[\alpha(G(x\,),\alpha(y)\,] = 0.\forall x,y\in R$

$\alpha(G(x)\,)\ \alpha(y) - \alpha(y)\, \alpha(G(x)) = 0\, .$

$\alpha(G(x)\, y) - \alpha\, (yG(x)\,) = 0.$

$\alpha(G(x)\, y - yG(x)\ \ ) = 0.$

$\alpha\, [G(x),y] = 0.\ .\forall x,y\in R$

Since $\alpha\neq 0$ is an automorphism of R we get $[G(x),y] = 0.\ \forall x,y\in R$.

Hence G maps from R into Z(R). ◇

Next we deal with a prime*-ring R and semi-simple*-ring.

<u>Corollary 2.3:</u> Let R be a prime *-ring. If R admits a generalized (α,β) *- derivation F with an associated non-zero $(\alpha,\beta)^*$ - derivation d, then either F = 0 or R is commutative.

<u>Proof</u>: In the view of Theorem 1 we have

$F(x)[\alpha(y),\alpha(z)]= 0.\ \forall x,y,z\in R.$ (15)

Replacing y by yt in (15) we get $F(x)\, [\alpha(yt),\alpha(z)] = 0$

$F(x)\, [\alpha(y)\, \alpha(t),\alpha(z)\,] = 0.$

$=F(x)[\alpha(y),\alpha(z)]\alpha(t)+ F\, (x)\alpha(y)\, [\, \alpha(t),\alpha(z)] = 0.$ (16)

$=F\, (x)\, \alpha(y)\, [\alpha(t),\alpha(z)] = 0.\forall x,y,z\, \in R.$(By 15)

$= F\, (x)R\, [\, \alpha(t),\alpha(z)] = 0.\forall x,t,z\in R.$ (17)

Primness of R forces (17) to either $F\, (x) = 0$ or $[\, \alpha(t),\alpha(z)] = 0.\ \forall x,t,z,\in R.$

Consider $[\, \alpha(t),\alpha(z)] = 0.\forall t,z\in R.$

$\alpha(t)\, \alpha(z) - \alpha(z)\, \alpha(t) = 0\, .$

$\alpha(tz) - \alpha(zt) = 0.$

$\alpha\, (\, [t,z]\,) = 0.$

Since $\alpha\neq 0$ is an automorphism of R we get $[t, z] = 0.\ \forall t,z\in R.$

Hence either F = 0 or R is commutative.

<u>Corollary2.4:</u> Let R be a semi-simple *-ring.If R admits generalized (α,β) *- derivation F with an associated non-zero $(\alpha,\beta)^*$ - derivation d, the F maps from R into Z(R).

<u>Theorem 2.5:</u> Let R be semi- prime *-ring . If R admits a generalized reverse(α,β) *- derivation F with an associated non-zero reverse $(\alpha,\beta)^*$ - derivation d,then $[d(x),z\,] = 0$.

Proof: $F(xy)=F(y)\alpha(x^*)+\beta(y)d(x)\ \forall x,y\, \in R$ (18)

Replacing x by xz in (18) and using the fact that d is $(\alpha\beta)$*-derivation we get

$F(xzy) = F(y)\, \alpha\, (\, (xz)^*)+\beta(y)\, d(xz).$

$= F(y)\, \alpha(z^*x^*) + \beta(y)\ \ (\, d(z)\, \alpha(x^*)+ \beta(z)d(x)\,)$

$=F(y)\alpha(z^*)\alpha(x^*)+\beta(y)d(z)\alpha(x^*)+\beta(y)\beta(z)d(x).$ (19)

On the other hand

$F(xzy)=F(x(zy))$

$= F(zy)\alpha(x^*)+\beta(zy)d(x).$

$=F(y)\alpha(z^*)\alpha(x^*)+\beta(y)d(z)\alpha(x^*)+\beta(z)\beta(y)d(x).$ (By (18) (20)

Comparing (19) and (20) we get

$[\beta(y),\beta(z)]d(x)=0 \forall x,y,z\in R$. (21)

Replacing y by d(x) y in (21) we get

$[\beta(d(x) y), \beta(z)] d(x) = 0.$

$=[\beta(d(x) \beta(y), \beta(z)]d(x) = 0.$

$= \beta(d(x) [\beta(y), \beta(z)] d(x) +[\beta(d(x), \beta(z)] \beta(y)d(x) = 0.$

$= [\beta(d(x),\beta(z)] \beta(y)d(x) = 0. \forall x,y,z \in R$ (By (21) (22)

Right multiplication of (22) by β(z d(x)) we get

$[\beta(d(x), \beta(z)] \beta(y)d(x) \beta(zd(x)) = 0.$ (23)

Right multiplication of (22) by β (d(x)z) we get

$[\beta(d(x),\beta(z)] \beta(y)d(x) \beta(d(x)z) = 0.$ (24)

Comparing (23) and (24) we get

$[\beta(d(x)), \beta(z)] \beta(y)d(x) [\beta(d(x)), \beta(z)] = 0. \forall x,y,z \in R$

$[\beta(d(x)), \beta(z)] R [\beta(d(x)), \beta(z)] = 0.$ (25)

By semi-primeness of R , (25) reduces to

$[\beta(d(x)), \beta(z)] = 0. \forall x,z \in R$

$= \beta(d(x)) \beta(z) - \beta(z) \beta(d(x)) = 0.$

$= \beta(d(x) z) - \beta(zd(x)) = 0.$

$= \beta[d(x), z] = 0$

since $\beta\neq 0$ we get $[d(x), z] = 0 \forall x,z \in R$

Hence the theorem.

The Next corollary states that a non-commutative prime*-ring R admits generalized reverse (α,β) *- derivation F then F is a right α*- centralizer.

<u>Corollary 2.6</u>: Let R be a non-commutative prime *-ring. If R admits a generalized reverse (α,β) *- derivation F with an associated non-zero reverse (α,β) *- derivation d, then F is a right α*- centralizer

<u>Proof</u>: By theorem (2.5) we have

$[\beta(y)\beta(z)]d(x)=0 \forall x,y,z\in R$ (26)

Replacing y by xy in (26) we get

$[\beta(xy), \beta(z)]d(x) = 0. \forall x,y,z \in R$.

$=[\beta(x) \beta(y), \beta(z)]d(x) = 0.$

$= \beta(x)[\beta(y), \beta(z)]d(x)+[\beta(x), \beta(z)] \beta(y) d(x) = 0.$

$=[\beta(x), \beta(z)] \beta(y) d(x) = 0. \forall x,y,z \in R$ (By (26)

$= [\beta(x), \beta(z)] R d(x) = 0. \forall x,z \in R$

The primeness of R forces the above equation to either $[\beta(x), \beta(z)] = 0$ or $d(x) = 0$.

Consider $[\beta(x), \beta(z)] = 0 = \beta(x) \beta(z)- \beta(z) \beta(x)$

$= \beta\,(xz) - \beta(zx).$

$= \beta\,[x,z]$

ie $\beta\,[x,z] = 0$. Since $\beta \neq 0$ is endomorphism of R we get $[x,z] = 0$

Therefore either $[x,z] = 0$ or $d(x)=0$.

Put $U=\{x \in R / [x,z] = 0\}$ and $V = \{x \in R / d(x) = 0\}$.

Then U and V are additive subgroups of R such that $U \cup V = R$.

But R cannot be union of two of its proper subgroups we find that

U=R or V =R.

If U = R then $[x, z] = 0\ \forall x, z \in R$ and hence R is commutative, a contradiction.

On the other hand if V =R then $d(x) = 0. \forall x \in R$ then d = 0.

$F(xy) = F(y)\, \alpha(x^*)$

F is right α *-centralizer.

<u>Corollary 2.7</u>: Let R be semi-prime *-ring. If R admits non-zero reverse $(\alpha, \beta)^*$ - derivation d, then d maps from R into Z(R).

Proof: choose F = d in the proof of theorem 3.

3. Conclusion

The motivation of the result for which a generalized $(\alpha,\beta)^*$-derivation F which is mapping from a 2-torsion free semi-prime*- ring R to the center Z(R) plays a key role in this total article. Hence it is proved some other results regarding a prime*-ring R admits a generalized $(\alpha, \beta)^*$- derivation F Which is equal to zero or R is commutative, a non-commutative prime*-ring R admits a generalized reverse $(\alpha, \beta)^*$- derivation F then F is right α^*- centralizer.

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