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On 2- Normed Space Valued Orlicz Space $\ell_\infty((S, \|.,.\|),\Phi, \bar{w})$ of Bounded Sequences and its Topological Structure

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Abstract: *The aim of this paper is to introduce and study a new class* $\ell_\infty((S, \|. , .\|), \Phi, \bar{w})$ *of 2-normed space valued sequences using Orlicz function as a generalization of the basic space* ℓ_∞ *of bounded complex sequences studied in Functional Analysis. Besides the investigation of conditions pertaining to the containment relation of the class* $\ell_\infty((S, \|.,.\|), \Phi, \bar{w})$ *in terms of different* $\bar{w}$ *, our primary interest is to explore the linear space structures of the class* $\ell_\infty((S, \|. , .\|), \Phi, \bar{w})$ *with some topological properties.*

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1. Introduction

So far, a good number of research works have been done on various types of algebraic and topological properties of sequence spaces using Orlicz function as the generalization of various well known sequence spaces for instances, (see [1], [2] ,[3], [4], [5],[6],[7], [8],[9],[10] and [11].

Definition 1.1 A function $\Phi : [0, \infty) \to [0, \infty)$ is called an *Orlicz function*, if it is continuous, non-decreasing and convex with $\Phi(0) = 0$, $\Phi(x) > 0$ for $x > 0$ and $\Phi(x) \to \infty$ as $x \to \infty$. An Orlicz function Φ can be represented in the following integral form

$$\Phi(x) = \int_0^x q(t)\, dt$$

where q, known as the kernel of Φ, is right-differentiable for $t \geq 0$, $q(0) = 0$, $q(t) > 0$ for $t > 0$, q is non decreasing, and $q(t) \to \infty$ as $t \to \infty$, (see, Krasnosel'skiî and Rutickiî [12]).

Definition 1.2 An Orlicz function Φ is said to satisfy Δ_2 - condition for all values of x, if there exists a constant $K > 0$ such that

$$\Phi(2x) \leq K\,\Phi(x), \text{ for all } x \geq 0.$$

The Δ_2-condition is equivalent to the satisfaction of the inequality $\Phi(Lx) \leq K L\, \Phi(x)$ for all values of x for which $L > 1$, (see, Krasnosel'skiî and Rutickiî [12]).

The notion of 2–normed space was initially introduced by S. GÄahler [13] as an interesting linear generalization of a normed linear space, which was subsequently studied in [14], [15], [16] and many others. Recently a lot of activities have been started by many researchers to study this concept in different directions, for instances,(see, ,[11], [17], [18], [19]).

Definition 1.3 Let S be a vector space of dimension greater than 1 over K, the field of real or complex numbers. A 2 - *norm* on S is a real valued function $\|. , .\|$ on $S \times S$ satisfying the following conditions:

(i) $\| \xi, \eta \| \geq 0$ and $\| \xi, \eta \| = 0$ if and only if ξ and η are linearly dependent;

(ii) $\| \xi, \eta \| = \| \eta, \xi \|$, for all $\xi, \eta \in S$;

(iii) $\| \alpha\xi, \eta \| = |\alpha| \, \|\xi, \eta\|$, where $\alpha \in K$ and $\xi, \eta \in S$;

(iv) $\| \xi_1 + \xi_2, \eta \| \leq \| \xi_1, \eta \| + \| \xi_2, \eta \|$ for all ξ_1, ξ_2 and $\eta \in S$.

The pair $(S, \|. , .\|)$ is called a 2–normed space.Recall that $(S, \|. , .\|)$ is a 2-Banach space if every Cauchy sequence in S is convergent to some s_0 in S.

Geometrically, a 2-norm function generalizes the concept of area function of parallelogram spanned by the two associated vectors, see [18].

For example, consider $S = \boldsymbol{R}^2$, being equipped with $\| \bar{\xi}, \bar{\eta} \| = | \xi_1 \eta_2 - \xi_2 \eta_1|$, where $\bar{\xi} = (\xi_1, \xi_2)$ and $\bar{\eta} = (\eta_1 , \eta_2)$. Then $(S, \|. , .\|)$ forms a 2–normed space and $\| \bar{\xi} , \bar{\eta} \|$ represents the area of the parallelogram spanned by the two associated vectors $\bar{\xi}$ and $\bar{\eta}$.

Analogously, if $S = \boldsymbol{R}^3$ and define the function $\|. , .\|$ on $S \times S$ by

$$\| \bar{\xi}, \bar{\eta} \| = \left| \operatorname{Det} \begin{pmatrix} i & j & k \\ \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{pmatrix} \right|$$

where $\bar{\xi} = (\xi_1, \xi_2, \xi_3)$ and $\bar{\eta} = (\eta_1, \eta_2, \eta_3)$. Obviously $(S, \|. , .\|)$ forms a 2–normed space.

Definition 1.4. Let S be a normed space over $\boldsymbol{C}$, the field of complex numbers. Let $\omega(S)$ denotes the linear space of all sequences $\bar{\xi} = < \xi_k >$ with $\xi_k \in S$, $k \geq 1$ with usual coordinate wise operations i.e., $\bar{\xi} + \bar{\eta} = < \xi_k + \eta_k >$ and $\alpha \bar{\xi} = < \alpha\xi_k >$,for each $\bar{\xi} , \bar{\eta} \in \omega (S)$ and $\alpha \in \boldsymbol{C}$.

We shall denote $\omega(\boldsymbol{C})$ by ω . Further, $\bar{\lambda} = < \lambda_k > \in \omega$ and $\bar{\xi} = < \xi_k > \in \omega (S)$ we shall write $\bar{\lambda} \bar{\xi} = < \lambda_k \xi_k >$. By a vector valued sequence space we mean a linear subspace of $\omega (S)$.

Definition 1.5 Lindenstrauss and Tzafriri [20] used the idea of Orlicz function to construct the sequence space ℓ_Φ of scalars $< \xi_k >$ such that

$$\ell_\Phi = \left\{ \bar{\xi} = < \xi_k > \in \omega \; : \; \sum_{k=1}^{\infty} \Phi \left(\frac{| \xi_k |}{r} \right) < \infty \text{ for some } r > 0 \right\}.$$

The space ℓ_Φ with the norm

$$\| \bar{\xi} \|_\Phi = \inf \left\{ r > 0 : \sum_{k=1}^{\infty} \Phi \left(\frac{|\xi_k|}{r} \right) \leq 1 \right\}$$

becomes a Banach space which is called an *Orlicz sequence space*.The space ℓ_Φ is closely related to the space ℓ_p which is an Orlicz sequence space with $\Phi (t) = t^p : 1 \leq p < \infty$.

Definition 1.6. A sequence space S is said to be *solid* if $\bar{\xi} = < \xi_k > \in S$ and $\bar{\alpha} = < \alpha_k >$ a sequence of scalars with $|\alpha_k| \leq 1$, for all $k \geq 1$, then $\bar{\alpha}\bar{\xi} = < \alpha_k \xi_k > \in S$.

2. THE CLASS ℓ_∞ $((S, \|. , .\|), \Phi, w)$

Let $\bar{w} = < w_k >$ and $\bar{v} = < v_k >$ be any sequences of strictly positive real numbers .Let $(S, \|.,.\|)$ be the 2- normed space over the field $\boldsymbol{C}$ of complex numbers and θ denote the zero element of S. Let $\omega(S)$ denote the linear space of all sequences $\bar{\xi} = < \xi_k >$ with $\xi_k \in S$, $k \geq 1$ with usual coordinate wise operations.We now introduce the following class of 2- normed space *S*–valued sequences using Orlicz function Φ.

$\ell_\infty((S, \|. , .\|), \Phi, \bar{w}) = \{\bar{\xi} = <\xi_k> \in \omega(S)$ such that for some $r > 0$ satisfying

$$\sup_k \Phi\left(\frac{1}{r}\|\xi_k , s\|^{w_k}\right) < \infty, \text{ for each } s \in S\}. \quad (2.1)$$

Further, when $w_k = 1$ for all *k*, then $\ell_\infty((S, \|. , .\|), \Phi, \bar{w})$ will be denoted by $\ell_\infty((S, \|. , .\|), \Phi)$

If in the definition of $\ell_\infty((S, \|. , .\|), \Phi, \bar{w})$ in (2.1), the phrase 'for some $r > 0$' is replaced by 'for every $r > 0$' then we denote this subclass by $\overline{\ell_\infty}((S, \|. , .\|), \Phi, \bar{w})$. Thus

$\overline{\ell_\infty}((S, \|. , .\|), \Phi, \bar{w}) = \{\bar{\xi} = <\xi_k> \in \omega(S)$ such that for every $r > 0$ satisfying

$$\sup_k \Phi\left(\frac{1}{r}\|\xi_k , s\|^{w_k}\right) < \infty, \text{ for each } s \in S\}. \quad (2.2)$$

3. CONTAINMENT RELATIONS

In this section, we investigate some inclusion relations between the classes $\ell_\infty((S, \|. , .\|), \Phi, \bar{w})$ arising in terms of different $\bar{w}$. Throughout, we shall denote

$\sup w_k = L$ for all $k \geq 1$ and for scalar α, $M[\alpha] = \max(1, |\alpha|)$.

But when the sequences w_k and v_k occur, then to distinguish *L* we use the notations $L(w)$ and $L(v)$ respectively.

Theorem 3.1: $\ell_\infty((S, \|. , .\|), \Phi, \bar{w}) \subset \ell_\infty((S, \|. , .\|), \Phi, \bar{v})$ if $\lim\sup_k \frac{v_k}{w_k}$ is finite.

Proof: Assume that $\lim\sup_k \frac{v_k}{w_k} < \infty$. Then there exists a positive constant *d* such that $v_k < d\, w_k$ for all sufficiently large values of *k*. Let $\bar{\xi} = <\xi_k> \in \ell_\infty((S, \|. , .\|), \Phi, \bar{w})$.Then for some $r > 0$,

$$\sup_k \Phi\left(\frac{1}{r}\|\xi_k , s\|^{w_k}\right) < \infty, \text{ for each } s \in S.$$

Hence we can find a positive real number η satisfying

$$\Phi\left(\frac{1}{r}\|\xi_k , s\|^{w_k}\right) \leq \Phi\left(\frac{\eta}{r}\right)$$

and therefore $\|\xi_k, s\|^{w_k} < \eta$ for each $s \in S$ and for all sufficiently large values of *k*.

Since $v_k < d\, w_k$ for all sufficiently large values of *k* and so if $\|\xi_k, s\| \leq 1$, for each $s \in S$, then $\|\xi_k, s\|^{v_k} \leq 1$; and on the other hand if $\|\xi_k, s\| > 1$ for each $s \in S$, then $\|\xi_k, s\|^{v_k} < \|\xi_k, s\|^{d\, w_k} < \eta^d$.

Therefore $\|\xi_k, s\|^{v_k} \leq A[\eta^d]$, for each $s \in S$ and for all sufficiently large values of *k*.This shows that for each $s \in S$ and for all sufficiently large values of *k*,

$$\Phi\left(\frac{1}{r}\|\xi_k , s\|^{v_k}\right) \leq \Phi\left(\frac{A[\eta^d]}{r}\right)$$

and therefore $\sup_k \Phi\left(\frac{1}{r}\|\xi_k , s\|^{v_k}\right) < \infty$.

Thus $\bar{\xi} \in \ell_\infty((S, \|. , .\|), \Phi, \bar{v})$ and hence

$\ell_\infty((S, \|. , .\|), \Phi, \bar{w}) \subset \ell_\infty((S, \|. , .\|), \Phi, \bar{v})$.

Theorem 3.2: If $\ell_\infty((S, \|.,.\|), \Phi, \bar{w}) \subset \ell_\infty((S, \|.,.\|), \Phi, \bar{v})$ then $\limsup_k \frac{v_k}{w_k}$ is finite.

Proof: Suppose that the inclusion holds but $\limsup_k \frac{v_k}{w_k} = \infty$. Then there exists a sequence $< k(n) >$ of positive integers such that $k(n+1) > k(n) \geq 1, n \geq 1$, for which

$v_{k(n)} > n\, w_{k(n)}$, for all $n \geq 1$. (3.1)

Now, corresponding to $t \in S$ and $t \neq \theta$, we define a sequence $\bar{\xi} = < \xi_k >$ by

$$\xi_k = \begin{cases} 2^{1/w_{k(n)}}\, t, & \text{if } k = k(n),\ n \geq 1 \text{ and} \\ \theta, & \text{otherwise.} \end{cases} \tag{3.2}$$

Let $r > 0$. Then for each $s \in S$, we have

$$\sup_k \Phi\left(\frac{1}{r}\|\xi_k, s\|^{w_k}\right) = \sup_n \Phi\left(\frac{1}{r}\|2^{1/w_{k(n)}}t, s\|^{w_{k(n)}}\right)$$

$$= \sup_n \Phi\left(\frac{2}{r}\|t, s\|^{w_{k(n)}}\right)$$

$$\leq \Phi\left(\frac{2M[\|t, s\|^{L(w)}]}{r}\right) < \infty.$$

This shows that $\bar{\xi} \in \ell_\infty((S, \|.,.\|), \Phi, \bar{w})$. But on the other hand, let us choose $s \in S$ such that $\|t, s\| = 1$. Then in view of (3.1) and (3.2) ,we have

$$\sup_k \Phi\left(\frac{1}{r}\|\xi_k, s\|^{v_k}\right) = \sup_n \Phi\left(\frac{1}{r}\|2^{1/w_{k(n)}}t, s\|^{v_{k(n)}}\right)$$

$$\geq \sup_n \Phi\left(\frac{2^n}{r}\right) = \infty.$$

This shows that $\bar{\xi} \notin \ell_\infty((S, \|.,.\|), \Phi, \bar{v})$, contradicting our assumption.This completes the proof.

Combining the Theorems 3.1 and 3.2, we have

Theorem 3.3: $\ell_\infty((S, \|.,.\|), \Phi, \bar{w}) \subset \ell_\infty((S, \|.,.\|), \Phi, \bar{v})$ if and only if $\limsup_k \frac{v_k}{w_k} < \infty$.

Theorem 3.4: $\ell_\infty((S, \|.,.\|), \Phi, \bar{v}) \subset \ell_\infty((S, \|.,.\|), \Phi, \bar{w})$ **if and only if $\liminf_k \frac{v_k}{w_k} > 0$.**

Proof:

Assume that $\liminf_k \frac{v_k}{w_k} > 0$. So that there exists a $m > 0$ such that $v_k > m\, w_k$, for all sufficiently large values of k. Then analogous to the Theorem 3.1, sufficiency part follows.

Conversely, suppose that the inclusion holds but $\liminf_k \frac{v_k}{w_k} = 0$.Then we can find a sequence $< k(n) >$ of positive integers such that $k(n+1) > k(n) \geq 1$, $n \geq 1$, for which

$n\, v_{k(n)} < w_{k(n)}$, for each $n \geq 1$. (3.3)

Now, taking $t \in S$, $t \neq \theta$ and define a sequence $\bar{\xi} = < \xi_k >$ by

$$\xi_k = \begin{cases} 2^{1/v_{k(n)}}\, t, & \text{if } k = k(n),\ n \geq 1 \text{ and} \\ \theta, & \text{otherwise.} \end{cases} \tag{3.4}$$

Let $r > 0$. Then for each $s \in S$, we have

$$\sup_k \Phi\left(\frac{1}{r}\|\xi_k, s\|^{v_k}\right) = \sup_n \Phi\left(\frac{1}{r}\|2^{1/v_{k(n)}} t, s\|^{v_{k(n)}}\right)$$

$$= \sup_n \Phi\left(\frac{2}{r}\|t, s\|^{v_{k(n)}}\right)$$

$$\leq \Phi\left(\frac{2M[\|t, s\|^{L(v)}]}{r}\right) < \infty.$$

This shows that $\bar{\xi} \in \ell_\infty((S, \|.,.\|), \Phi, \bar{v})$. But on the other hand, let us choose $s \in S$ such that $\|t, s\| = 1$. Then in view of (3.3) and (3.4) ,we have

$$\sup_k \Phi\left(\frac{1}{r}\|\xi_k, s\|^{w_k}\right) = \sup_n \Phi\left(\frac{1}{r}\|2^{1/v_{k(n)}} t, s\|^{w_{k(n)}}\right)$$

$$\geq \sup_n \Phi\left(\frac{2^n}{r}\right) = \infty.$$

This shows that $\bar{\xi} \notin \ell_\infty((S, \|.,.\|), \Phi, \bar{w})$, a contradiction. The proof is now complete.

On combining the Theorems 3.3 and 3.4, one obtain

Theorem 3.5: $\ell_\infty((S, \|.,.\|), \Phi, \bar{w}) = \ell_\infty((S, \|.,.\|), \Phi, \bar{v})$ if and only if

$0 < \liminf_k \frac{v_k}{w_k} \leq \limsup_k \frac{v_k}{w_k} < \infty$.

Corollary 3.6:

(i) $\ell_\infty((S, \|.,.\|), \Phi) \subset \ell_\infty((S, \|.,.\|), \Phi, \bar{w})$ **if and only if** $\limsup_k w_k < \infty$;

(ii) $\ell_\infty((S, \|.,.\|), \Phi, \bar{w}) \subset \ell_\infty((S, \|.,.\|), \Phi)$ **if and only if** $\liminf_k w_k > 0$;**and**

(iii) $\ell_\infty((S, \|.,.\|), \Phi, \bar{w}) = \ell_\infty((S, \|.,.\|), \Phi)$ **if and only if** $0 < \liminf_k w_k \leq \limsup_k w_k < \infty$.

Proof: The proof follows by using $w_k = 1$ for all k and $\bar{v}$ *is* replaced by $\bar{w}$ in Theorems 3.3 , 3.4 and 3.5 respectively. In the following example, we show that $\ell_\infty((S, \|.,.\|), \Phi, \bar{w})$ may strictly be contained in $\ell_\infty((S, \|.,.\|), \Phi, \bar{v})$ in spite of the satisfaction of the condition of Theorem 3.1.

Example 3.7:

Let $(S, \|.,.\|)$ be a 2- normed space and for $t \in S$, $t \neq \theta$,we define a sequence $\bar{\xi} = <\xi_k>$ in S by

$\xi_k = k^k t$, if $k = 1, 2, 3, \ldots$

Further, let $w_k = k^{-1}$, if k is odd integer, $w_k = k^{-2}$, if k is even integer, $v_k = k^{-2}$ for all values of k.

Further, $\frac{v_k}{w_k} = \frac{1}{k}$, if k is odd integer, $\frac{v_k}{w_k} = 1$, if k is even integer. Therefore $\limsup_k \frac{v_k}{w_k} = 1 < \infty$. Hence the condition of Theorem 3.1 is satisfied.

Let $r > 0$. Then for each $s \in S$, we have

$$\sup_k \Phi\left(\frac{1}{r}\|\xi_k, s\|^{v_k}\right) = \sup_k \Phi\left(\frac{1}{r}\|k^k t, s\|^{1/k^2}\right)$$

$$= \sup_k \Phi\left(\frac{(k)^{1/k}}{r}\|t, s\|^{1/k^2}\right)$$

$$\leq \sup_k \Phi\left(\frac{1}{r}\|\, t, s\,\|^{1/k^2}\right)$$

$$\leq \Phi\left(\frac{M[\,\|\,t, s\,\|\,]}{r}\right) < \infty.$$

This shows that $\bar{\xi} \in \ell_\infty((S, \|.\,,.\|), \Phi, \bar{v})$. But **on** the other hand, let us choose $s \in S$ such that $\|\, t, s\,\| = 1$. Then for each odd integer k, we have

$$\Phi\left(\frac{1}{r}\|\,\xi_k, s\|^{w_k}\right) = \Phi\left(\frac{\|k^k t, s\,\|^{1/k}}{r}\right)$$

$$= \Phi\left(\frac{k\,\|\, t, s\,\|^{1/k}}{r}\right) = \Phi\left(\frac{k}{r}\right),$$

which implies that $\bar{\xi} \notin \ell_\infty((S, \|.\,,.\|), \Phi, \bar{w})$. Thus the containment of $\ell_\infty((S, \|.\,,.\|), \Phi, \bar{w})$ in $\ell_\infty((S, \|.\,,.\|), \Phi, \bar{v})$ is strict inspite of the satisfaction of the condition of Theorem 3.1.

4. Linear Space Structure of $\ell_\infty((S, \|.\,,.\|), \Phi, w)$

In this section, we shall investigate some results that characterize the linear space structure of the class $\ell_\infty((S, \|.\,,.\|), \Phi, \bar{w})$ with some topological properties. Throughout we take coordinate-wise operations of sequences over the field ***C*** of complex numbers i.e.,for $\bar{\xi} = <\xi_k>$ and $\bar{\eta} = <\eta_k>$ and scalar α,

$$\bar{\xi} + \bar{\eta} = <\xi_k + \eta_k> \quad \text{and} \quad \alpha\,\bar{\xi} = <\alpha\xi_k>$$

And we see below that $\ell_\infty((S, \|.\,,.\|), \Phi, \bar{w})$ forms a linear space over ***C***. Moreover, we use frequently

$$|\, a + b\,|^{w_k} \leq M[2^{L-1}]\,\{|a|^{w_k} + |b|^{w_k}\},$$

where $a, b \in$ **C**, $0 < \sup_k w_k = L < \infty$, $M[\alpha] = \max\{1, |\alpha|\}$ for scalar α and $Q = M[2^{L-1}]$.

Theorem 4.1: $\ell_\infty((S, \|.\,,.\|), \Phi, \bar{w})$ **forms a linear space over C if $\sup_k w_k$ is finite.**

Proof: Suppose that $\sup_k w_k < \infty$, $\bar{\xi} = <\xi_k>$ and $\bar{\eta} = <\eta_k> \in \ell_\infty((S, \|.\,,.\|), \Phi, \bar{w})$ and $\alpha, \beta \in$ ***C***. Then there exist $r_1 > 0$ and $r_2 > 0$ such that for each $s \in S$, we have

$$\sup_k \Phi\left(\frac{1}{r_1}\|\,\xi_k, s\|^{w_k}\right) < \infty \quad \text{and} \quad \sup_k \Phi\left(\frac{1}{r_2}\|\,\eta_k, s\|^{w_k}\right) < \infty.$$

We now choose $r > 0$ such that

$2\,Q\,r_1 M\,[|\alpha|^L] \leq r$ and $2\,Q\,r_2 M\,[\,|\beta|^L] \leq r$, where $Q = M[2^{L-1}]$.

For such r, using non decreasing and convex properties of Φ, we have

$$\Phi\left(\frac{1}{r}\|\,\alpha\,\xi_k + \beta\eta_k, s\|^{w_k}\right) \leq \Phi\left[\frac{1}{r}\left(Q\|\alpha\,\xi_k, s\,\|^{w_k} + Q\|\beta\eta_k, s\|^{w_k}\right)\right]$$

$$= \Phi\left[\frac{Q}{r}|\alpha|^{w_k}\|\,\xi_k, s\|^{w_k} + \frac{Q}{r}|\beta|^{w_k}\|\,\eta_k, s\|^{w_k}\right]$$

$$= \Phi\left[\frac{Q}{r}M[\,|\alpha|^L]\,\|\,\xi_k, s\|^{w_k} + \frac{Q}{r}M[\,|\beta|^L]\,\|\,\eta_k, s\|^{w_k}\right]$$

$$\leq \Phi\left[\frac{1}{2\,r_1}\|\,\xi_k, s\|^{w_k} + \frac{1}{2\,r_2}\|\,\eta_k, s\|^{w_k}\right]$$

$$\leq \frac{1}{2}\Phi\left(\frac{1}{r_1}\|\xi_k, s\|^{w_k}\right) + \frac{1}{2}\Phi\left(\frac{1}{r_2}\|\,\eta_k, s\|^{w_k}\right)$$

Thus,

$$\sup_k \Phi\left(\frac{1}{r}\| \alpha\, \xi_k + \beta\eta_k, s\|^{w_k} \right) \leq \frac{1}{2} \sup_k \Phi\left(\frac{1}{r_1}\|\xi_k, s\|^{w_k} \right) + \frac{1}{2} \sup_k \Phi\left(\frac{1}{r_2}\| \eta_k, s\|^{w_k} \right) < \infty,$$

for each $s \in S$ and hence $\alpha\bar{\xi} + \beta\bar{\eta} \in \ell_\infty ((S, \|. , .\|), \Phi, \bar{w})$. This implies that $\ell_\infty ((S, \|. , .\|), \Phi, \bar{w})$ forms a linear space over $\boldsymbol{C}$.

Theorem 4.2: If Φ satisfies the Δ_2 -condition, then

$\ell_\infty ((S, \|. , .\|), \Phi, \bar{w}) = \bar{\ell_\infty} ((S, \|. , .\|), \Phi, \bar{w})$.

Proof: To prove the theorem, it suffices to show that $\ell_\infty ((S, \|. , .\|), \Phi, \bar{w})$ is a subset of $\bar{\ell_\infty} ((S, \|. , .\|), \Phi, \bar{w})$ since the reverse inclusion follows by definition. Let $\bar{\xi} \in \ell_\infty ((S, \|. , .\|), \Phi, \bar{w})$. Then for some $r > 0$ and for each $s \in S$,

$$\sup_k \Phi\left(\frac{1}{r}\| \xi_k , s\|^{w_k} \right) < \infty.$$

Let us consider an arbitrary $r_1 > 0$.

If $r \leq r_1$, then obviously, we have

$$\sup_k \Phi\left(\frac{1}{r_1}\| \xi_k, s\|^{w_k} \right) \leq \sup_k \Phi\left(\frac{1}{r}\| \xi_k , s\|^{w_k} \right) < \infty,$$

for each $s \in S$. Hence we get $\bar{\xi} \in \bar{\ell_\infty} ((S, \|. , .\|), \Phi, \bar{w})$.

On the other hand, if $r > r_1$ then $\frac{r}{r_1} > 1$. In this case, using Δ_2 -condition of Φ ,we get

$$\sup_k \Phi\left(\frac{1}{r}\| \xi_k , s\|^{w_k} \right) = \sup_k \Phi\left(\frac{r}{r_1} . \frac{1}{r}\| \xi_k, s\|^{w_k} \right)$$

$$\leq K. \frac{r}{r_1} \sup_k \Phi\left(\frac{1}{r}\| \xi_k , s\|^{w_k} \right) < \infty,$$

for each $s \in S$, where K is the number involved in Δ_2 - condition. This proves that

$\bar{\xi} \in \bar{\ell_\infty} ((S, \|. , .\|), \Phi, \bar{w})$.

Corollary 4.3: If Φ satisfies the Δ_2 -condition, then $\bar{\ell_\infty} ((S, \|. , .\|), \Phi, \bar{w})$ forms a linear space over $\boldsymbol{C}$.

Proof: Proof follows from using Theorems 4.1 and 4.2.

Theorem 4.4: The space $\ell_\infty ((S, \|. , .\|), \Phi , \bar{w})$ forms a solid.

Proof: Let $\bar{\xi} = < \xi_k > \in \ell_\infty ((S, \|. , .\|), \Phi , \bar{w})$. So that

$$\sup_k \Phi\left(\frac{1}{r}\| \xi_k , s\|^{w_k} \right) < \infty,$$ for some $r > 0$ and for each $s \in S$.

Let $< \rho_k >$ be a sequence of scalars satisfying $|\rho_k| \leq 1$ for all $k \geq 1$. Using non decreasing property of Φ , we have

$$\sup_k \Phi\left(\frac{1}{r}\| \rho_k\, \xi_k, s\|^{w_k} \right) = \sup_k \Phi\left(\frac{1}{r}|\rho_k|^{w_k} \| \xi_k, s\|^{w_k} \right)$$

$$\leq \sup_k \Phi\left(\frac{1}{r}\| \xi_k , s\|^{w_k} \right) < \infty,$$

For each $s \in S$. This shows that $< \rho_k \xi_k > \in \ell_\infty((S, \|.\,,.\|), \Phi, \overline{w})$ and hence $\ell_\infty((S, \|.\,,.\|), \Phi, \overline{w})$ is solid.

5. Conclusion

In this paper, we have examined some conditions that characterize the linear topological structures and containment relations on 2- normed space valued Orlicz Space of bounded sequences. In fact, these results can be used for further generalization to investigate other properties of the spaces of 2- normed space valued bounded sequences using Orlicz function.

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