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Group Inverses of Con-s-k-EP Matrices

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Abstract: *In this paper, the existence of the group inverse for con-s-k-EP matrices under certain condition is derived.*

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1. Introduction

Let c_{nxn} be the space of nxn complex matrices of order n. let C_n be the space of all complex n tuples. For Aϵc_{nxn} . Let $\bar{A}$, A^T, A^*, A^S , $\bar{A}^S$, $A^\dagger$, R(A), N(A) and $\rho(A)$ denote the conjugate, transpose, conjugate transpose, secondary transpose, conjugate secondary transpose, Moore Penrose inverse range space, null space and rank of A respectively. A solution X of the equation AXA = A is called generalized inverse of A and is denoted by A^- . If A ϵ c_{nxn} then the unique solution of the equations A XA =A , XAX = X, $[AX]^*$ = AX , $\mathrm{XA}^* = \mathrm{XA}$ **[3]** is called the Moore-Penrose inverse of A and is denoted by $A^\dagger$. A matrix A is called Con-s-$k - EP_r$ if ρ A $=$ r and N(A) = N(A^TVK) (or) R(A)=R(KVA^T). Throughout this paper let "k" be the fixed product of disjoint transposition in S_n= { 1,2,….n} and k be the associated permutation matrix .

Let us define the function $\boldsymbol{k}$ (x)= $x_{k\,1}, x_{k\,2}, ..., x_{k\,n}$. A matrix A = $(a_{ij})\ \epsilon\ c_{nxn}$ is s-k-symmetric if a_{ij}= $a_{n-k(j)+1,n-k(i)+1}$ for i, j = 1,2,…..n . A matrix A ϵ c_{nxn} is said to be Con-s-k-EP if it satisfies the condition A_x = 0 $<=>$ $A^s\ k\ (x)$ = 0 or equivalently N(A) =N(A^TVK). In addition to that A is con-s-k-EP $<=>$ KVA is con-EP or AVK is con-EP and A is con-s-k-EP$<=>$ A^T is con-s-k-EP$_r$ moreover A is said to be Con-s-k-EP$_r$ if A is con-s-k-EP and of rank r. For further properties of con-s-k-EP matrices one may refer **[2]**.

Theorem 2 (p.163) [1]

Let $A \in C_{n\times n}$. Then A is $EP \Leftrightarrow A^{\#} = A^{\dagger}$ when $A^{\#}$ exists.

It is well known that, for an con-EP matrix, group inverse exists and coincides with its Moore-Penrose inverse. However, this is not the case for a con-s-k-EP matrix. For example, consider $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ for k=(1,2)(3), the associated permutation matrix $K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

A is con-s-k-EP$_1$ matrix, But $A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \rho(A) = \rho(A^2)$.

Therefore by **Theorem 2 ,** group inverse $A^{\#}$ does not exists for A. Here it is proved that for a con-s-k-EP matrix A, if the group inverse exists then it is also a con-s-k-EP matrix.

Theorem 2.1.1

Let $A \in C_{n\times n}$ be con-s-k-EP$_r$ and $\rho(A) = \rho(A^2)$. Then $A^{\#}$ exists and is con-s-k-EP$_r$.

Proof

Since, $\rho(A) = \rho(A^2)$, by **Theorem 2** , $A^{\#}$ exists for A. To show that $A^{\#}$ is con-s-k-EP$_r$, it is enough to prove that $R(A^{\#}) = R(KV(A^{\#})^T)$.

Since, $AA^{\#} = A^{\#}A$, we have, $R(A) = R(AA^{\#})$

$$= R(A^{\#}A)$$

$$= R(A^{\#})$$

$$AA^{\#}A = A \Rightarrow A^T = A^T(A^{\#})^T A^T$$

$$\Rightarrow KVA^T = KVA^T A^{\#T} A^T$$

Therefore, $R(KVA^T) = R(KVA^T A^{\#T} A^T)$

$$= R(KVA^T A^{\#T})$$

$$= R(KV(A^{\#}A)^T)$$

$$= R(KV(AA^{\#})^T)$$

$$= R(KVA^{\#T} A^T)$$

$$= R(KVA^{\#T})$$

Now, A is con-s-k-EP$_r$ $\Leftrightarrow R(A) = R(KVA^T)$ and $\rho(A) = r$ $\Leftrightarrow R(A^T) = R(KV(A^{\#})^T)$ and $\rho(A) = \rho(A^{\#}) = r \Leftrightarrow A^{\#}$ is con-s-k-EP$_r$.

Remark 2.1.2

In the above Theorem the condition that $\rho(A) = \rho(A^2)$ is essential.

Example 2.1.3

Let $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for k = (1,2)(3), the associated permutation matrix

$K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. $KVA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is EP$_1 \Rightarrow A$ is con-s-k-EP$_1$. Since $A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \rho(A^2) = 0.$

That is, $\rho(A) \neq \rho(A^2).$ Hence $A^{\#}$ does not exists for a con-s-k-EP matrix $A.$ Thus, for a con-s-k-EP matrix A, if $A^{\#}$ exists then it is also con-s-k-EP$_r$.

Theorem 2.1.4

For $A \in C_{n\times n}$ if $A^{\#}$ exists then, A is con-s-k-EP $\Leftrightarrow (KVA)^{\#} = A^{\dagger}VK.$

Proof

A is con-s-k-EP $\Leftrightarrow KVA$ is con-EP **(by Theorem (2.11) [2])**

$\Leftrightarrow (KVA)^{\#} = (KVA)^{\dagger}$ **(by Theorem 2)**

$\Leftrightarrow (KVA)^{\#} = A^{\dagger}VK$ **(by Theorem (2.1.12))**

Theorem 2.1.5

For $A \in C_{n\times n}$, A is con-s-k-EP$_r$ $\Leftrightarrow A^{\dagger} = KV$ (polynomial in AVK)
$=$(polynomial in KVA)$VK.$

Proof

It is clear that if $(KVA)^{\dagger} = f(KVA)$ for some scalar polynomial $f(x)$ then KVA commutes with $(KVA)^{\dagger}$.

$$\Rightarrow (KVA)(KVA)^{\dagger} = (KVA)^{\dagger}(KVA)$$
$$\Rightarrow (KVA)(A^{\dagger}VK) = (A^{\dagger}VK)(KVA)$$
$$\Rightarrow KVAA^{\dagger}VK = A^{\dagger}VKKVA$$
$$\Rightarrow KVAA^{\dagger}VK = A^{\dagger}A$$
$$\Rightarrow KVAA^{\dagger} = A^{\dagger}AKV$$
$\Rightarrow A$ is con-s-k-EP$_r$

(ByTheorem(2.11)[2]) Conversely, let A be con-s-k-EP$_r$, then $KVAA^{\dagger} = A^{\dagger}AKV$ and $KVA^{\dagger}A = AA^{\dagger}KV.$ Now, we will prove that $A^{\dagger}$ can be expressed as KV (polynomial in AVK) and (polynomial in KVA)$VK.$ Let $(KVA)^s + \lambda_1(KVA)^{s+1} + ... + \lambda_q(KVA)^{s+q} = 0$ be the minimal polynomial of $KVA.$ Then $s = 0 (or) s = 1.$ For suppose $s \geq 2,$ then

$$(KVA)^{\dagger}[(KVA)^s + \lambda_1(KVA)^{s+1} + ... + \lambda_q(KVA)^{s+q}] = 0 .$$

Hence,

$$[(KVA)(KVA)^{\dagger}(KVA)](KVA)^{s-2} + \lambda_1[(KVA)(KVA)^{\dagger}(KVA)](KVA)^{s-1} + ...$$
$$+\lambda_q[(KVA)(KVA)^{\dagger}(KVA)](KVA)^{s+q-2} = 0$$

That is, $(KVA)^{s-1} + \lambda_1(KVA)^s + ... + \lambda_q(KVA)^{s+q-1} = 0$ this is a contradiction.

If $s = 0,$ then

$$(KVA)^{\dagger} = (KVA)^{-1} = -\lambda_1 I - \lambda_2 (KVA) .... - \lambda_q (KVA)^{q-1}$$

$$A^{\dagger} VK = A^{-1} VK = -\lambda_1 I - \lambda_2 (KVA) .... - \lambda_q (KVA)^{q-1}$$

$$A^{\dagger} = A^{-1} = [-\lambda_1 I - \lambda_2 (KVA) .... - \lambda_q (KVA)^{q-1}] KV$$

$$= (\text{polynomial in } KVA) KV$$

Thus, $A^{\dagger} =$ (polynomial in KVA) KV. If $s = 1$, then

$(KVA)^{\dagger} [KVA + \lambda_1 (KVA)^2 + ... + \lambda_q (KVA)^{q+1}] = 0$ and it follows that $(KVA)^{\dagger}(KVA) = -\lambda_1 (KVA) - \lambda_2 (KVA)^2 .... - \lambda_q (KVA)^q$ is a polynomial in A. However, $(KVA)^{\dagger} = [(KVA)^{\dagger}(KVA)](KVA)^{\dagger}$

$$= -\lambda_1 (KVA)^{\dagger}(KVA) - \lambda_2 (KVA) .... - \lambda_q (KVA)^{q-1}]$$

$$A^{\dagger} VK = -\lambda_1 A^{\dagger} VKKVA - \lambda_2 (KVA) .... - \lambda_q (KVA)^{q-1}$$

$$A^{\dagger} = -\lambda_1 A^{\dagger} AKV - \lambda_2 (KVA) KV .... - \lambda_q (KVA)^{q-1} KV$$

$$= [-\lambda_1 A^{\dagger} A - \lambda_2 (KVA) .... - \lambda_q (KVA)^{q-1}] KV$$

$$= (\text{polynomial in } KVA) KV.$$

Thus, $A^{\dagger} =$ (polynomial in KVA) KV.

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