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# On Tridiagonal Conjugate K-Normal Matrices

**K. Gunasekaran**
Department of Mathematics
Government Arts College (Autonomous),
Kumbakonam, Tamilnadu, India

**K. Arumugam**
Department of Mathematics,
A.V.C College (Autonomous),
Mayiladuthurai, Tamilnadu, India
*karumugam83@gmail.com*

**Abstract:** *Conjugate k-normal matrices play some role in the theory of k-unitary congruence as conventional k-normal matrices; do with respect to k-unitary similarities. Naturally, the properties of both matrix classes are fairly similar up to the distinction between the congruence and similarity. However in certain respects, conjugate k-normal matrices differ substantially from k-normal ones. Our goal in this paper is to indicate one of such distinctions. It is shown that none of the familiar characterizations of k-normal matrices having the irreducible tridiagonal from has a natural counterpart in the case of conjugate of k-normal matrices*

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## 1. Introduction

A matrix $A \in C_{n\times n}$ is said to be conjugate k-normal if

$$AA^*K = \overline{KA^*}A. \tag{1}$$

This matrix class plays the same role in the theory of k-unitary congruences as conventional k-normal matrices do with respect to k-unitary similarities. Accordingly the properties of both matrix classes are fairly similar up to the distinction between congruence Similarity. However in certain respects, conjugate k-normal matrices substantially differ from k-normal ones. Our goal in this paper is to indicate one of such distinctions that concerns matrices having a tridiagonal form.

A tridiagonal matrix

$$\begin{bmatrix} \alpha_1\beta_1 & \cdots & \cdots \\ \gamma_2\alpha_2\beta_3 & \cdots & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & \cdots & \alpha_{n-1}\beta_n \\ \cdots & \cdots & \gamma_n\alpha_n \end{bmatrix} \tag{2}$$

is said to be irreducible if $\beta_2.......\beta_n \neq 0$ (3)

and $\gamma_2.......\gamma_n \neq 0.$ (4)

For a k-normal A, in equalities (3) & (4) are implications of each other; therefore, irreducibility can be characterized by any one of these inequalities.

There exist several descriptions of k-normal matrices having the irreducible tridiagonal form. One of these descriptions is based on a well known characteristic property of k-normal matrices; namely, a matrix $A \in C_{n\times n}$ is k-normal if and only if it's k-hermitian adjoint $A^*$ is a polynomial in A. More over in the representation

$$A^* = f(A), \tag{5}$$

one can choose ‘f’ to be a polynomial with a degree less than n.

**Proposition 1.1:** Irreducible matrix (2) is k-normal if and only if $A^*$ is a linear polynomial in A.

The following description is an easy corollary of proposition 1.

**Proposition 1.2:** Irreducible matrix (2) is k-normal if and only if

$$A = e^{i\theta} H + \alpha I_n, \tag{6}$$

where $\phi \in R,\ \alpha \in C$, and H is a k-hermitian matrix. In particular, if A is real, then A is either k-symmetric or has the form

$$A = T + \alpha I_n, \tag{7}$$

where $\alpha \in R$ and T is a skew k-symmetric matrix.

One more description can be derived from representation (6).

**Proposition 1.3:** Irreducible matrix (2) is k-normal if and only if its spectrum belongs to a line.

In what follows, we show that none of these descriptions has a natural counterpart in the case of conjugate k-normal matrices. k-symmetric and skew k-symmetric matrices are congruent analogues of k-hermitian and skew k-hermitian matrices, respectively. However, an arbitrary conjugate k-normal matrix of form (2) cannot, in general, be reduced to these two special cases. This follows from the description of such matrices given in section 2. However, any matrix of form (2) still has a property that links it to k-symmetric and skew k-symmetric matrices; namely, any leading principal submatrix of matrix (2) is itself conjugate k-normal. We emphasize that, for a general conjugate k-normal matrix, an analogous property of principal submatrices is an exception rather than a rule.

Instead of k-eigen values, the theory of k-unitary congruence deals with con-k-eigen values, we show in section 3 that, in general, the conspectrum of a conjugate k- normal matrix of form (2) cannot be located on a line.

There exists a characterization of conjugate k-normal matrices that is similar to representation (5) (see [1]). However, one must allow for polynomials with a degree greater then 1in such a representation. This follows from the discussion in section 4.

## 2. Tridiagonal Conjugate K-Normal Matrices

Hereafter, matrix (2) is assumed to be irreducible. Moreover, without loss of generality, we can assume $\beta_2, \beta_3, \ldots, \beta_n$ to be real positive scalars. Indeed, performing for matrix (2) the congruence transformation $A \to \widetilde{A} = DAD$ with a diagonal k-unitary matrix,

$$D = diag\{1, d_2, \ldots, d_n\},\ d_j = e^{i\delta_j} \qquad j = 2, 3, \ldots, n\ ,$$

we have $\tilde{\alpha}_{12} = \beta_2 d_2$ , $\tilde{\alpha}_{j,j+1} = \beta_{j+1} d_j d_{j+1}$ $\quad j = 2, 3, \ldots, n-1$.

Setting $\delta_2 = -\arg \beta_2$ , $\delta_{j+1} = -\arg \beta_{j+1} - \delta_{j,}\ j = 2, 3, \ldots, n-1$ ,

we obtain a matrix $\widetilde{A}$ with the entries in portions (1,2),(2,3)…….(n-1,n).

Denote by $A_{n-1}$ the leading principal submatrix that is obtained by deleting the last row and the least column in A.

**Lemma 2.1:** $A_{n-1}$ is a conjugate k-normal matrix.

**Proof:**

Equating the last k-diagonal entries of the two matrices in relation (1), we see that

$$|\gamma_n| = \beta_n . \tag{8}$$

Equating the leading principal submatrices of order n-1 in (1), we have

$$A_{n-1}A_{n-1}^{*}K+\beta_n^2 e_{n-1}e_{n-1}^T=\overline{KA_{n-1}^{*}A_{n-1}}+|\gamma_n|^2 e_{n-1}e_{n-1}^T$$

Here $e_{n-1}$ in the last coordinate column vector in the space $\mathbb{C}^{n-1}$. Equalities (8) & (9) prove the lemma.

**Corollary 2.2:** All the leading principal sub matrices of a conjugate k-normal matrix A of form (2) are also conjugate k-normal.

**Remarks 2.3:** A similar assertion is valid for trailing submatrices, that is, for submatrices counted off the right lower corner of A. Moreover, any principal submatrix lying at the intersection of successive rows and columns of matrix (2) is conjugate k-normal.

Now, we equate in (1) the entries in the positions (n-2,n) and (n-1,n), which gives

$$\begin{aligned}&\beta_{n-1}\overline{\gamma_n}=\beta_n\gamma_{n-1}\\&\alpha_{n-1}\overline{\gamma_n}+\overline{\alpha_n}\beta_n=\alpha_{n-1}\beta_n+\overline{\alpha_n}\gamma_n\end{aligned} \tag{10}$$

or

$$\alpha_{n-1}(\overline{\gamma_n}-\beta_n)=\overline{\alpha_n}(\gamma_n-\beta_n) \tag{11}$$

Using lemma 1 and its corollary recursively, we obtain the relations

$$\beta_{j-1}\overline{\gamma_j}=\beta_j\gamma_{j-1},\ \ j=3,4,...,n \tag{12}$$

$$\alpha_{j-1}(\overline{\gamma_j}-\beta_j)=\overline{\alpha_j}(\gamma_j-\beta_j),\ \ j=2,3,...,n \tag{13}$$

According to (12), a choice of $\beta_2,\beta_3,...,\beta_n$ and $\gamma_n$ uniquely determines $\gamma_2\,\gamma_3,...,\gamma_{n-1}$. Note that $\gamma_n$ must obey the condition (8). If $\gamma_n=\beta_n$ , then (12) implies the inequalities $\gamma_j=\beta_j$ , $j=2,3,...,n-1$. In this case, A is k-symmetric matrix and relations (13) impose no limitations on its k-diagonal entries $\alpha_1,...,\alpha_n$. If $\gamma_n=-\beta_n$, then the equalities

$$\gamma_j=-\beta_j \quad \text{j=2,3,….. n-1} \tag{14}$$

are derived from (12) and the equalities

$$\alpha_{j-1}=\overline{\alpha_j} \quad \text{j=2,3,….. n-1} \tag{15}$$

are derived from (13).

Thus, the choice of $\alpha_n$ determines the entries k-diagonal of A.

Finally, assume that $\gamma_n=\beta_n\,e^{i\phi},\quad \phi\in(-\pi,\pi),\ \phi\neq 0$ (16)

Relation (2) yields $\gamma_{n-1}=\beta_{n-1}e^{-i\phi},\gamma_{n-2}=\beta_{n-2}e^{i\phi},\gamma_{n-3}=\beta_{n-3}e^{-i\phi}$ (17)

Similarly to the preceding case, all the k-diagonal entries have the same modulus.

Define $\psi$ by the formula $\psi=\arg(e^{i\phi}-1)$ (18)

Chasing $\alpha_n$ , we find from (13) that

$$\alpha_{n-1}=\overline{\alpha_n}e^{i2\psi},\alpha_{n-2}=\overline{\alpha_{n-1}}e^{-i2\psi}=\alpha_n e^{-i4\psi},\alpha_{n-3}=\overline{\alpha_n}e^{i6\psi} \tag{19}$$

For instance, if n= 3 and $\phi = \frac{\pi}{2}$ , we have $\psi = \arg(i-1) = \frac{3}{4}\pi$ and $\gamma_2 = -\beta_2 i, \alpha_2 = -i\overline{\alpha_3}$, $\alpha_1 = -\alpha_3$. In the case described by relations (16)-(19), conjugate k-normal matrices of the form (2) cannot be reduced to k-symmetric or skew k-symmetric matrices.

## 3. On the Multiplicity of Con K- Eigenvalues

If Irreducible matrix (2) is k-normal, then all of its k-eigen values are simple, which follows from the relation rank $(A - zI_n) \geq n-1 \quad forall\, z \in \mathbb{C}$ .

This consideration is inapplicable to non k-normal matrices. For instance, the Jordan block $J_n(0)$ with zero on the main diagonal has the rank n-1 and , at the same time, an k-eigen value of multiplicity n. In general, conjugate k-normal matrices are not k-normal. Moreover, the con k-eigen values rather than k-eigen values are invariants of k-unitary congruences. We recall their definition as given in [2].

With a matrix $A \in C_{n\times n}$ , we associate the matrices

$$A_L = A\overline{A} \tag{20}$$

and

$$A_R = \overline{A}A. \tag{21}$$

Although, in general, the products AB and BA do not need to be similar, $\overline{A}A$ is always similar to $A\overline{A}$ (see 3, section 4.6, problem 9). Therefore, in the subsequent discussion of the spectral properties of these matrices, it suffices to consider only one of them, say, $A_L$ .

The spectrum of $A_L$ has two remarkable properties

(1) It is k-symmetric about the real axis. More ever, the k-eigen values $\lambda$ and $\overline{\lambda}$ have the same multiplicity.

(2) The negative k-eigen values of $A_L$ (if any) are necessarily of even algebraic multiplicity.

The proof s' of these assertions can be found in [1, section 4.6, problem 5-7].

Let $\lambda(A_L) = \{\lambda_1, ..., \lambda_n\}$ (22)

be the spectrum of $A_L$ .

**Definition 3.1:** The con k-eigen values of A are the n scalars $\mu_1, ..., \mu_n$ introduced as follows.

(a) If $\lambda_i \in \lambda(A_L)$ does not lie on the negative real semi axis, the corresponding con k-eigen values $\mu_i$ is defined as the square root of $\lambda_i$ with a non negative real part.

$$\mu_i = \lambda_i^{1/2}, \quad \mathrm{Re}\,\mu_i \geq 0. \tag{23}$$

The multiplicity of $\mu_i$ is set equal to that of $\lambda_i$.

(b) With a real negative $\lambda_i \in \lambda(A_L)$, we associate two conjugate purely imaginary con k-eigen values $\mu_i = \pm\lambda_i^{1/2}$ (24)

The multiplicity of each con k-eigen values is set equal to half the multiplicity of $\lambda_i$.

The set $c_\lambda(A) = \{\mu_1 .......\mu_n\}$ (25)

is called the con spectrum of A.

For a k-symmetric A, we have $\overline{A} = A^*, A_L = A^* A$, thus, the con k-eigen values of A are identical to its singular values. If A is skew k-symmetric, then

$$\overline{A} = -A^*, A_L = -A^* A$$

As noted above, every negative k-eigen values $\lambda$ of $A_L$ has an even multiplicity. It gives rise to two purely imaginary con k-eigen values $\mu = \pm i\sqrt{|\lambda|}$ of half the multiplicity. The most important property of k-normal matrices is every matrix of this class can be transformed into a k-diagonal matrix by a proper k-unitary similarity transformation. The k-diagonal entries of the transformed matrix are the k-eigen values of A. This spectral theorem for k-normal matrices has the following counterpart in the theory of k-unitary congruences [4, 5].

**Theorem 3.2:** Every conjugate k-normal matrix $A \in \mathbb{C}_{n\times n}$ can be brought by a proper k-unitary congruence transformation to a block k-diagonal from with k-diagonal blocks of order 1 and 2. The 1 by 1 blocks are non negative con k- eigen values of A. Each 2 by 2 block corresponds to a pair of complex conjugate con k-eigen values $\mu_j = \rho_j e^{i\theta_j}$, $\overline{\mu_i}$ and has the form

$$\begin{pmatrix} 0 & e_j \\ e_j e^{-2\theta_j} & o \end{pmatrix} \tag{26}$$

Or
$$\begin{pmatrix} 0 & \mu_j \\ \overline{\mu_j} & o \end{pmatrix} \tag{27}$$

Every matrix $A \in \mathbb{C}_{n\times n}$ can be represented in the form

$$A = S + T, \tag{28}$$

where $$S = \frac{1}{2}(A + A^T), \quad T = \frac{1}{2}(A - A^T). \tag{29}$$

Matrices (29) are called the real and imaginary parts of A, respectively. For a conjugate k-normal matrix A, decomposition (28) and (29) has a number of special properties. We need the property stated in the following proposition (sec [6]).

**Theorem 3.3:** Let A be a conjugate k-normal matrix with decomposition (28), (29). Then, the con k-eigen values of S (respectively, T) are the real (respectively, imaginary) parts of the con k-eigen values of A.

**Corollary 3.4:** If a conjugate k-normal matrix A has a pair of complex con k-eigen values $\mu = \sigma + it, \overline{\mu}$, then $\sigma$ is multiple con k-eigen values of $S = \frac{1}{2}(A + A^T)$ . The number of real con k-eigen values of A is equal to the multiplicity of zero as a con k-eigen values of $T = \frac{1}{2}(A - A^T)$

We return to conjugate k-normal matrices of form (2) that satisfy relation (16)-(19). In this case, it is easy to see that matrices (29) are tridiagonal along with A

**Lemma 3.5:** The multiplicity of each con k-eigen values of S (respectively, T) is almost two.

**Proof:**

For definiteness, we consider S. The con k-eigen values of this matrix are non negative scalars whose squares are the conventional k-eigen values of the five k-diagonal matrix,

$$S_L = \overline{S} S = S^* S .$$

The irreducibility of S ensures that all the entries of $S_L$ lying on the k-diagonal i – j = 2 are non zero. It follows that $rank(S_L - xI_n) \geq n-2$ for all $x \in \mathbb{R}$. Therefore, the multiplicity of each k-eigen values of the k-hermitian matrix $S_L$ is at most two.

**Corollary 3.6:** A conjugate k-normal matrix A described by relation (16)-(17) has at most two real con k-eigen values. All the pairs of conjugate con k-eigen values of A have distinct real parts. The corresponding con k-eigen values of $S = \frac{1}{2}(A + A^T)$ are double. By contrast, all the nonzero con k-eigen values of $T = \frac{1}{2}(A - A^T)$ are simple. These assertions are direct implications of lemma 2, theorem 2, and corollary 2.

Corollary 3 makes obvious the following ultimate conclusion, the conspectrum of a conjugate k-normal matrix described by relations (16)-(19) cannot be located on a line in the complex plane.

We conclude this section by a small illustration of the few given below. It is easy to verify that tridiagonal matrix $A = \begin{pmatrix} 1 & 0 & 1+i \\ 0 & 1 & 1+i \\ -1+i & -1+i & 1 \end{pmatrix}$ is conjugate k-normal.

Its symmetric part $s = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & i \\ i & i & 1 \end{pmatrix}$ has the simple con k-eigenvalue 1 and the double con k-eigenvalue $\sqrt{3}$.

The nonzero con k-eigenvalues of the skew k-symmetric part $s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$ are equal to $\pm i\sqrt{2}$ . Thus, $c\lambda(A) = \{1, \sqrt{3} + i\sqrt{2}, \sqrt{3} - i\sqrt{2}\}$.

## 4. On the Representation of the Transposed Matrix

Returning of representation (5), we call how proposition 1 can be proved. Assume that the degree ‘t’ of the polynomial ‘f’ in (5) is greater than one. Then, it is easy to see that the entries of A lying on the k-diagonals i – j = t and j – i = t must be non zero. This, however, contradicts the fact that f(A) must be the tridiagonal matrix $A^*$

The following assertion proved in [1] can be considered an analogue of representation (5) for conjugate k-normal matrices.

**Theorem 4.1:**

A matrix $A \in \mathbb{C}_{n\times n}$ is conjugate k-normal if and only if

$$A^T = f(A_R)A = Af(A_L) \tag{30}$$

for a polynomial f with real co-efficient. This polynomial can be chosen so that its degree is less than n. suppose that $A \neq 0$ and the polynomial f in (30) has a zero degree, that is $A^T = \alpha A$. A comparison of the norms of the left and right hand sides reveals that $|\alpha| = 1$. Furthermore, it is easy to verify that the equality $A^T = e^{i\phi}A$ is possible only for $\phi = \pi t, t \in \mathbb{Z}$. Thus, in the case, A is either k-symmetric or skew k-symmetric. Now, we show that, for any a conjugate k-normal matrix described by relations (16)-(19), the degree ‘t’ of ‘f ’ in representation (30) must be atleast (n/2). Indeed, assuming the contrary, that is, $0 < t \leq (\frac{n}{2}) - 1 \leq \frac{n-2}{2}$, we observe that $A(A_L)^t$ is the only monomial in the matrix $A\,f(A_L)$ that has the non zero entries on the k-diagonal i – j = 1 + 2t (which does not exceed n-1) and on the k-diagonal j – i = 1 + 2t . The same k-diagonal must be nonzero in $A\,f(A_L)$. This contradicts the fact that $A\,f(A_L)$ must be k-tridiagonal matrix $A^T$ .

## References


[1] Ikramov Kh.D, “Schur Inequality for coneigenvalues and conjugate- Normal Matrices” Vestn. Mosk.Gos.Univ., Ser.15: Vechisl.Mat.Kibern., No 3, 3-6 (1997).

[2] George A , Ikramov Kh.D, Matushkina E.V, and W.P.Tang, “ On a QR-like Algorithm for some structured Eigen value Problems”, SIAM J.Matrix Anal, Appl. 16, 1107-1126(1995).

[3] Horn R.A and Johnson C.R, Matrix Analysis (Cambridge univ. press, Combridge, 1985; Mir, Moscow,1989).

[4] Winger E.P, “ Normal form of Antiunitary Operators”, J.Math.Phys.II,

[5] 409-413(1960).

[6] Vujic M, Herbut F and Vujicic G, “ Canonical Forms for Matrices under unitary Congruences Transformation I ; Conjugate –Normal Matrices”, SIAM J. Appl.Math.23, 225-238(1972).

[7] Ikramov Kh.d,” On the Coneigenvalues and singular values of a Complex Square Matrix”, Zap. Nauchn,Semin .POMI 334,111-120(2006).