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Topological Conjugacy and the Chaotic Nature of the Family of Mappings $f_c(x) = x^2 - x + c$

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Abstract: *Many authors have explored the chaotic behavior of many families of mappings like the Tent family, Quadratic family, the logistic family, etc. The analysis of the nature of the logistic family*

$F_\mu(x) = \mu x(1 - x)$ has played an important role in the development of the subject of dynamical system and chaos. The chaotic nature of the family of mappings $f_c(x) = x^2 - x + c$ through the period doubling cascade has already been proved by Kulkarni P. R. and Borkar V. C. In this paper, the topological conjugacy of the family of mappings $f_c(x) = x^2 - x + c$ with the mapping σ has been established and thereby, the chaotic nature of $f_c(x) = x^2 - x + c$ in the sense of Devaney R. L. has been proved.

Keywords: *chaos, dynamical system, fixed points, orbits, periodic points, stability, topological conjugacy*

1. Introduction

Nearly in last thirty years, there has been a rapid development in the theory of dynamical systems and chaos. Computers have contributed a lot in this development. In this section we define some preliminary notions and state some fundamental results concerned. Many authors like Alligood [1], Devaney [2], Scheinerman [3] have defined the notion of dynamical systems and chaos. However, we take the most general definition given by Scheinerman [3].

1.1 Dynamical System

A dynamical system consists of a state vector $x \in R^n$ which is a list of numbers and may change as the time passes and a function $f : R^n \to R^n$, where the set R^n is called as the set of states or the state space. Given a state vector $x \in R^n$, the function f describes the rule by means of which the state vector x changes with time. The two types of dynamical systems *viz.* discrete and continuous with a variety of examples are given by [1], [2], [3] and [4]. In this paper, we will consider only one dimensional discrete dynamical system $f_c(x) = x^2 - x + c$.

1.2 Iterations of a Function

Let $f : S \to S$, $S \subseteq R$, be a given dynamical system. Iterations of the function f means the compositions of f with itself. Thus the first iteration of f at a point x of its domain means $f(x)$ itself. The second iteration of f at x is the composition $(f \circ f)(x) = f(f(x))$. It is also denoted by f^2.

Thus $f^2(x) = f(f(x))$.

In general, the k^{th} iteration of f at a point x is the k times composition of f with itself at the point x, denoted by $f^k(x)$. For more details, refer [2], [3], [7].

1.3 Fixed Points and Periodic Points

A point x is said to be a fixed point of a function f if $f(x) = x$. It is clear that if x is a fixed point of f, then $f^n(x) = x$ for all $n \in Z^+$, where is Z^+ the set of positive integers.

A point x_0 is said to be a periodic point with period n if $f^n(x_0) = x_0$ for some

$n \in Z^+$. It is clear that if x_0 is periodic with period n, then it is periodic with period 2n, 3n, 4n,...The smallest n, in this case, is called as the prime period of the orbit. Thus x_0 is a periodic point with period n of f if it is a fixed point of f^n. Refer [2], [3], [7].

1.4 Attracting and Repelling Fixed Points

Let p be a fixed point of a dynamical system $f : S \to S$, $S \subseteq R$.

(1) We say that p is an attracting fixed point or a sink of f if there is some neighborhood of p such that all points in this neighborhood are attracted towards p. In other words, p is a sink if there exists an epsilon neighborhood $N_\epsilon(p) = \{x \in S: |x - p| < \epsilon\}$ such that $\lim_{n\to\infty} f^n(x) = p$ for all $x \in N_\epsilon(p)$.

(2) We say that p is a repelling fixed point or a source of f if there is some neighborhood $N_\epsilon(p)$ of p such that each x in $N_\epsilon(p)$ except for p maps outside of $N_\epsilon(p)$. In other words, p is a source if there exists an epsilon neighborhood such that $|f^n(x) - p| > \epsilon$ for infinitely many values of positive integers n. Refer [2], [3], [7].

1.5 Hyperbolic Periodic Points

A periodic point p of a mapping f with prime period n is said to be hyperbolic

if $|(f^n)'(p)| \neq 1$. Refer [2], [3], [7].

1.6 Theorem

Let $f : [a, b] \to R$ be a differentiable function, where f' be continuous and p be a hyperbolic fixed point of f. If $|f'(p)| < 1$, then p is an attracting fixed point of f.

Proof: Refer [2], [3], [7].

1.7 Theorem

Let $f : [a, b] \to R$ be a differentiable function, where f' be continuous and p be a hyperbolic fixed point of f. If $|f'(p)| > 1$, then p is a repelling fixed point of f.

Proof: Refer [2], [3], [7].

1.8 Neutral Fixed Point

A fixed point p of a differentiable function f is said to be a neutral fixed point if $|f'(p)| = 1$.

1.9 Attracting and Repelling Periodic Point

Let p be a periodic point of period n of a function f. Then p is said to be an attracting periodic point or a repelling periodic point according as it is an attracting or a repelling fixed point of the n^{th} iterate f^n.

1.10 Theorem

Let $f :[a, b] \to R$ be a differentiable function, where f' be continuous and p be a periodic point of f with period n. Then the periodic orbit of p is attracting or repelling according as $|(f^n)'(p)| < 1$ or $|(f^n)'(p)| > 1$.

1.11 Theorem

Let p be a neutral fixed point of a function f.

(i) If $f''(p) > 0$, then p is weakly attracting from the left and weakly repelling from the right.

(ii) If $f''(p) < 0$, then p is weakly repelling from the left and weakly attracting from the right.

Let p be a neutral fixed point of f with If $f''(p) = 0$.

(iii) If $f'''(p) > 0$ then p is weakly repelling.

(iv) If $f'''(p) < 0$, then p is weakly attracting. (Refer [1],[2],[3])

2. Chaos

The term *chaos* is used when there is a type of randomness or uncertainty in a particular thing. Chaotic dynamics is the study of such randomness. There are many definitions of chaos given by different authors including measure theoretic notions, topological concepts, etc. We will use the topological ideas and define chaos in accordance with the definition given by Devaney, R. L. [2]. However, before going to the definition of chaos, we need to have a review of the some of the ideas used in it.

2.1 Sensitive Dependence on Initial Conditions

Let I be an interval in R. A mapping $f : I \to I$ has sensitive dependence on initial conditions if there exists a $\delta > 0$ such that, for every x in I and for every neighborhood N of x, there is some $y \in N$ and some positive integer n such that $| f^n(x) - f^n(y) | > \delta$.

In other words, f has sensitive dependence on initial conditions at x if for every

ϵ-neighborhood $N_\epsilon(x)$ of x, there is a point $y \epsilon N_\epsilon(x)$ and a δ such that $| f^n(x) - f^n(y) | > \delta$. If f has a sensitive dependence on initial conditions on each x in I, then we say that f has a sensitive dependence on initial conditions on I.

As an example, the Tent function T given by

$$T(x) = \begin{cases} 2x & \text{for } 0 \le x \le 1/2 \\ 2(1-x) & \text{for } 1/2 < x \le 1 \end{cases}$$

has sensitive dependence on initial conditions since it can be proved that after 10 iterations, the iterates of 1/3 and 0.333 are farther than 1/2 apart. Also, the logistic family $F_\mu(x) = \mu x(1 - x)$ has sensitive dependence on initial conditions on the set Λ for $\mu > 2 + \sqrt{5}$. Refer Devaney[2].

2.2 Topological Transitivity

Let I be an interval in R. A mapping $f : I \to I$ is said to be topologically transitive if there exists a pair U, V of open sets in I and a positive integer n such that $f^n(U) \cap V \neq \varphi$.

A topologically transitive mapping has points which eventually move under iterations from one arbitrarily small neighborhood to every other neighborhood. For example, it can be verified that the tent mapping T is topologically transitive on the interval [0, 1]. Refer Devaney[2].

2.3 Orbit and Seed

Let $f : S \to S$, $S \subseteq R$, be a given dynamical system. Given an initial point $x_0 \in S$, the orbit of x_0 under f is the sequence of iterates x_0, $x_1 = f(x_0)$, $x_2 = f^2(x_0)$, $x_3 = f^3(x_0)$, ..., $x_n = f^n(x_0)$,.... In this case, the initial point x_0 is called as the seed of the orbit.

2.4 Dense Set

Let S be a subset of R. A real number x is said to be a limit point of the set S if there exists a sequence $\{x_n\}$ of points in S that converges to x. The set S together with all its limit points is called as the closure of S and is denoted by $\bar{S}$. A set $S \subseteq R$ is said to be dense in R if $\bar{S} = R$.

2.5 Chaos

Let V be a set. A mapping $F : V \to V$ is said to be chaotic on V if

1. F has sensitive dependence on initial conditions.
2. F is topologically transitive.
3. periodic orbits are dense in V. See[2]

For example, the logistic family $F_\mu(x) = \mu x(1 - x)$ is chaotic on the set Λ for $\mu > 2 + \sqrt{5}$. Also, the tent mapping T is chaotic on the interval [0, 1]. Refer Devaney[2].

Though there are three conditions to be satisfied by a mapping to be chaotic, there are certain relations in these conditions. Since topological transitivity and existence of dense orbits are topological properties, these are preserved under homeomorphisms. However, sensitive dependence on initial conditions is a metric property and is not preserved under homeomorphisms in general. From the above definitions, it follows that if a map possesses a dense orbit, then it is topologically transitive. The converse holds in case of compact subsets of *R*. The authors J. Banks, J. Brooks and others [6] have proved that if a mapping is topologically transitive and has a dense orbit, then it has sensitive dependence on initial conditions.

3. TOPOLOGICAL CONJUGACY

The analysis of the family of mapping $f_c(x) = x^2 - x + c$ through its dynamics for different values of the real parameter *c* has been done by Kulkarni P. R. and Borkar V. C.[7]. In this analysis, the number and the nature of the fixed and periodic points of the family $f_c(x) = x^2 - x + c$ is explored in detail for different values of *c*. The chaotic behavior of the family of mappings $f_c(x) = x^2 - x + c$ has been proved through the so called *period doubling cascade*. In the current paper, we will prove that the mapping $f_c(x) = x^2 - x + c$ exhibits chaotic behavior using the notion of topological conjugacy. But before that, first we define topological conjugacy and prove some of the results associated with it.

3.1 Topological Conjugacy

Let $f : A \to A$ and $g : B \to B$ be two mappings. Then *f* and *g* are said to be topologically conjugate if there exists a homeomorphism $h : A \to B$ such that $h \circ f \circ h^{-1} = g$ or what amounts to the same thing, if $h \circ f = g \circ h$. In this case, the homeomorphism *h* is called as a topological conjugacy or we say that *f* and *g* are conjugate via the mapping *h*.

For example, the mapping $F_\mu(x) = \mu x(1-x)$ is topologically conjugate with the mapping $Q_c(x) = x^2 + c$ via the mapping $h(x) = -\mu x + \frac{\mu}{2}$, where $c = -\frac{\mu^2}{4} + \frac{\mu}{2}$. Also, it can be easily verified that the tent map *T* is conjugate with the map $F_\mu(x) = \mu x(1-x)$ for $\mu = 4$ via the conjugacy $h(x) = sin^2(\frac{\pi x}{2})$ defined on the interval [0, 1]. The importance of topological conjugacy lies in the following results.

3.2 Theorem

If *f* and *g* are topologically conjugate via mapping *h* and if *p* is a fixed point of *f*, then *h(p)* is a fixed point of *g*.

Proof: Since $h(p)=h(f(p))=(h \circ f)(p)=(g \circ h)(p)=g(h(p))$, it follows that *h(p)* is a fixed point of *g*.

Here we have a generalization.

3.3 Theorem

If *f* and *g* are topologically conjugate via mapping *h*, then $h \circ f^n = g^n \circ h$. Thus, if *f* and *g* are conjugate, the periodic points are carried into periodic points of the same period under conjugacy.

Proof: We prove the theorem by induction on *n*. As *f* and *g* are conjugate, $h \circ f = g \circ h$. Hence the result is true for $n = 1$. Assume that the result is true for some positive integer *m*, so that

$h \circ f^m = g^m \circ h$.

Now $h \circ f^{m+1} = (h \circ f) \circ f^m = (g \circ h) \circ f^m = g \circ (h \circ f^m) = g \circ (g^m \circ h) = g^{m+1} \circ h$.

Hence the result is true for $m+1$.

Thus by mathematical induction, the theorem is proved.

3.4 Theorem

Let $f : A \to A$ and $g : B \to B$ be topologically conjugate via a mapping $h : A \to B$. Then *f* is transitive if and only if *g* is transitive. That is, topological conjugacy preserves transitivity. **Proof:** First assume that *f* is transitive. Let *U* and *V* be non-empty open sets in *A* and *B* respectively. Then *h* being continuous, $h^{-1}(V)$ is open in *A*, and hence, it can be expressed as a union of open intervals in *A*. Moreover, *h* being onto, $h^{-1}(V)$ is non-empty. Let *I* be an open subset of $h^{-1}(V)$.

Similarly, let *J* be an open subset of the open set *h(U)*. As *f* is transitive, there exists a positive integer *n* and some *y* in *J* such that $(f^{\,n})(y) \epsilon I$. Taking $x = h(y)$, it follows that $x \epsilon U$ with $(g^{\,n})(x) = (g^{\,n})(h(y)) = h(f^n(y)) \epsilon\ h(I)$.

But as $h(I) \subseteq V$, it follows that $(g^{\,n})(x) \epsilon\ V$. This proves that *g* is also transitive. The converse follows with exactly the same arguments by interchanging the role of *f* and *g*. This completes the proof.

3.5 Theorem

If $h : A \rightarrow B$ is an onto continuous mapping, then the image under *h* of a set dense in *A* is a set dense in *B*.

Proof: The proof follows just by using the continuity of *h* and from the hypothesis that *h* is onto.

3.6 Set of Periodic Points

Let $Per_n(f)$ denote the set of periodic points of period *n* of the mapping *f*.

An immediate consequence of the theorems 3.3 and 3.5 is the following.

3.7 Theorem

Let *f* and *g* be topologically conjugate. Then $Per_n(f)$ is dense if and only if $Per_n(g)$ is dense.

From the definition of chaos and using the theorems 3.2 to 3.7, it follows that if two mappings are topologically conjugate, then they have exactly the same dynamics, that is, their behavior regarding the number and nature of the fixed and periodic points is the same. We summarize this as the following.

3.8 Theorem

Let *f* and *g* be topologically conjugate. Then *f* is chaotic if and only if *g* is chaotic.

3.9 The Sequence Space

The set $\Sigma_2 = \{\ s = (s_0\ s_1\ s_2 ...) : s_j = 0 \text{ or } 1\ \}$ is known as the sequence space on two symbols 0 and 1. By Devaney [2] it has been proved that the set Σ_2 is a metric space with respect to the metric *d* defined by $d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$,

where $s = (s_0\ s_1\ s_2 ...)$, $t = (t_0\ t_1\ t_2 ...) \in \Sigma_2$.

3.10 The Shift Map

The mapping $\sigma : \Sigma_2 \rightarrow \Sigma_2$ defined by $\sigma\,(s_0\ s_1\ s_2 ...) = (s_1\ s_2\ s_3 ...)$ is called as the shit mapping on the sequence space. By Devaney R. L.[2], the following results have been proved.

(i) The mapping σ is continuous.

(ii) $Card\ Per_n(\sigma) = 2^n$.

(iii) $Per(\sigma)$ is dense in Σ_2.

(iv) There exists a dense orbit for σ in Σ_2.

4. CHAOS IN $f_c(x) = x^2 - x + c$

The dynamics of the mapping $f_c(x) = x^2 - x + c$ has been analyzed in detail by Kulkarni P. R. and Borkar V. C. for the values of *c* greater than –1/2. When the value of *c* decreases below

–1/2, the dynamics of the mapping becomes more and more difficult to understand. We will attempt to study the behavior of $f_c(x) = x^2 - x + c$ for $c < -5/4$ and prove that f_c has chaotic nature using the concept of topological conjugacy. For a better understanding of the dynamics of f_c, we will use the concepts of Cantor's middle thirds set and the symbolic dynamics introduced by Devaney R. L.[2].

Consider the Figure 1 showing the graph of the function $f_{-3.3} = x^2 - x - 3.3$.

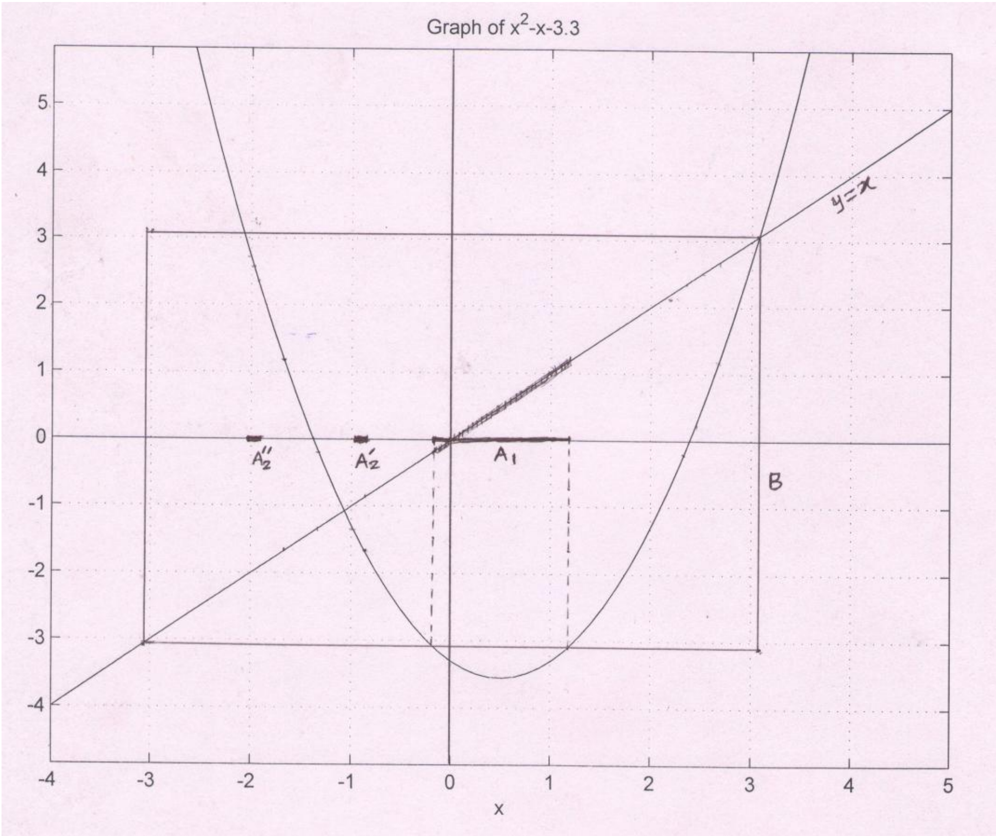

Figure 1

Recall that all the interesting dynamics of the mapping f_c occurs between the two fixed points $r_1 = 1 + \sqrt{1-c}$ and $r_2 = 1 - \sqrt{1-c}$ for all possible values of c. For the function $f_{-3.3}$, the surprising activity occurs in the interval $I = [-r_1, r_1] = [-3.0736, 3.0736]$. The function $f_{-3.3}$ is almost enclosed in the rectangular box B with vertices $(-3.0736, 3.0736)$, $(3.0736, -3.0736)$, $(3.0736, 3.0736)$ and $(-3.0736, -3.0736)$. The value $x \cong 3.0736$ is a fixed point of the function $f_{-3.3} = x^2 - x - 3.3$. We observe that the lower middle part of the graph is going out of the box B. Let A_1 be the part of the interval I for which the graph of the function is out of the box. This set $A_1$is the open interval $(-0.190185, 1.190185)$. The orbits of all x in the interval A_1 leave the interval I after the first iteration and escape to infinity. There are many such sets (like A_1) in the interval I where the orbits of the points in these intervals escape to infinity. Let us denote by Λ the set of all values of x in I whose orbits never go out of I. To be more specific, $\Lambda = \{x \in I : f_c^n(x) \in I \; \forall n \in Z^+\}$.Let Λ^c denote the complement of Λ in I *i.e.*

$\Lambda^c = I - \Lambda$. Then we have $A_1 \subset \Lambda^c$. There are infinitely many subsets of I whose union is Λ^c. Define $A_2 = \{x \in I : f_c(x) \in A_1, f_c^2(x) \notin I\}$. Thus A_2 is the set of points in I for which the first iteration lies in the set A_1 and all subsequent iterations leave I. Note that this set A_2 must be union of two open intervals, say A_2' and A_2'' as shown in the figure 1. The intervals A_2' and A_2'' can be determined from the graphical analysis[7]. Continuing the definition of the sets like A_1, A_2, let A_n be the set of points in I for which the $(n-1)^{th}$ iteration lies in the set A_{n-1} and the n^{th} iteration escapes from I. Then the set Λ^c is the set $\Lambda^c = A_1 \cup A_2 \cup A_3 \cup \ldots$. We observe that the pattern of the set Λ is exactly same to that of the Cantor's middle thirds set. In fact, the set Λ is a Cantor set in its own right. As $\Lambda \subseteq (\mathrm{I} - \mathrm{A}_1)$, the set $(\mathrm{I} - \mathrm{A}_1)$ is the union of two closed intervals, one to the left of the origin denoted I_0 and the other to the right side of the origin denoted I_1. Thus for each $x \in \Lambda$, $f_c^n(x)$ lies within I_0 or within I_1or jumps between I_0 and I_1.

4.1 Itinerary

For each $x \in \Lambda$, we define the itinerary of x as the infinite sequence $S(x) = s_0 s_1 s_2 \ldots$, where $s_j = 0$ if $f_c^j \in I_0$ and $s_j = 1$ if $f_c^j \in I_1$. Thus the itinerary of x is an infinite sequence of 0's and 1's so that $S(x) \in \Sigma_2$. We can consider S as a mapping from Λ to Σ_2.

Now we have enough material for proving our main result of this section.

4.2 Theorem

The family of mappings $f_c(x) = x^2 - x + c$ topologically conjugate to the shift map σ on Σ_2 via the itinerary S.

Proof: Let $x \in \Lambda$ be given. If $S(x) = s_0 s_1 s_2 \ldots$, then $\sigma(S(x)) = s_1 s_2 s_3 \ldots$.

As $x \in \Lambda$, it has some itinerary $(s_0 s_1 s_2 \ldots)$ so that $x \in I_{s_0}$.

Therefore, $f_c(x) \in I_{s_1}$,

$f_c^2(x) \in I_{s_2}$,

$f_c^3(x) \in I_{s_3}$, etc.

Since $I_{s_j} = I_0$ or $I_{s_j} = I_1$ for each s_j, we conclude that $S(f_c(x)) = s_1 s_2 s_3 \ldots$.

That is $\sigma \circ S = S \circ f_c$. This completes the proof. □

Since the shift map σ over the sequence space Σ_2 exhibits chaos, it follows that the family of mappings $f_c(x) = x^2 - x + c$ is chaotic over the set Λ for $c < -5/4$.

5. Conclusion

Dynamical systems and chaos have been a topic of great importance in last few years, specially chaos because chaos is the phenomenon observed almost everywhere in the nature. The analysis of the nature of many families of mappings have important applications in many fields. In this paper, we have proved chaos in the family $f_c(x) = x^2 - x + c$ and added one more family in the chaos theory, and thus stretched the surface of chaos. It would be interesting to find out what phenomenon in nature can be modeled in terms of this family of mappings and find the long term effects through the analysis made so far.

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