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Characterization of Uniquely Colorable and Perfect Graphs

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Abstract: *This paper studies the concepts of uniquely colorable graphs & Perfect graphs. The main results are*

1) *Every uniquely k-colorable graph is (k - 1)-connected.*
2) *If G is a uniquely k-colorable graph, then* $\delta(G) \geq k - l$.
3) *A maximal planar graph G of order 3 or more has chromatic number 3 if and only if G is Eulerian.*
4) *Every interval graph is perfect.*
5) *A graph G is chordal if and only if G can be obtained by identifying two complete. Sub graphs if the same order in two chordal graphs.*

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1. Uniquely Colorable Graphs

1.1 Definition

Suppose that G is a k-chromatic graph. Then every k-coloring of G produces a partition of $V(G)$ into k independent subsets (color classes). If every two k-colorings of G result in the same partition of $V(G)$ into color classes, then G is called **uniquely *k*-colorable** or simply **uniquely colorable**. Trivially, the complete graph K_n is uniquely colorable. In fact, every complete k-partite graph, $k \geq 2$, is uniquely colorable.

1.2 Theorem

In every k-coloring of a uniquely k-colorable graph G, where $k \geq 2$, the sub graph of G induced by the union of every two color classes of G is connected.

Proof. Assume, to the contrary, that there exist two color classes V_1 and V_2 in some k-coloring of G such that $H = G\,[V_1 \cup V_2]$ is disconnected. We may assume that the vertices in V_1 are colored 1 and those in V_2 are colored 2. Let H_1 and H_2 be two components of H. Interchanging the colors 1 and 2 of the vertices in H_1 produces a new partition of $V(G)$ into color classes, producing a contradiction.

1.3 Note

As a consequence of Theorem1.2, every uniquely k-colorable graph, $k \geq 2$, is connected. In fact, Gary Chartrand, and Dennis Paul Geller **[1]** showed that more can be said.

1.4 Theorem

Every uniquely k-colorable graph is $(k-1)$-connected.

Proof. The result is trivial for $k=1$ and, by Theorem 1.2, the result follows for $k=2$ as well. Hence we may assume that $k \geq 3$. Let G be a uniquely k-colorable graph, where $k \geq 3$. If $G=K_k$, then G is $(k - 1)$-connected; so we may assume, that G is not complete. Assume, to the contrary, that G is not $(k - 1)$-connected. Hence there exists a vertex cut W

of G with $|W| = k - 2$.

Let there be given a k-coloring of G. Consequently, there are at least two colors, say 1 and 2, not used to color any vertices of W. Let V_1 be the color class consisting of the vertices colored 1 and V_2 the set of the vertices colored 2. By Theorem1.2, $H = G[V_l \cup V_2]$ is connected, Hence H is a subgraph of some component G_1 of G-W. Let G_2 be another component of G-W. Assigning some vertex of G_2 the color 1 produces a new k-coloring of G that results in a new partition of $V(G)$ into color classes, contradicting our assumption that G is uniquely k-colorable.

1.5 Corollary

If G is a uniquely k-colorable graph, then $\delta(G) \geq k - l$.

Proof: Much of the interest in uniquely colorable graphs has been directed towards planar graphs. Since every complete graph is uniquely colorable, each complete graph K_n, $1 \leq n \leq 4$, is a uniquely colorable planar graph. Indeed, each complete graph K_n, $1 \leq n \leq 4$, is a uniquely colorable maximal planar graph. Since the complete 3 partite graph $K_{2,2,2}$ (the graph of the octahedron) is also uniquely colorable, $K_{2,2,2}$ is a uniquely 3-colorable maximal planar graph.(seeFigures1(a)).

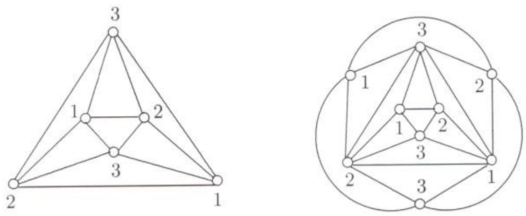

Figure (a). *uniquely 3-colorable maximal planar graphs*

The graph G in Figures 1(b) is also a uniquely 3-colorable maximal planar graph. The fact that the 3-colorable maximal planar graphs shown in Figure 1 are also uniquely colorable is not surprising, as Chartrand and Geller **[1]** observed.

1.6 Note

The two 3-colorable maximal planar graphs in Figure 1 have another property in common. There are both Eulerian. That this is a characteristic of all maximal planar 3-chromatic graphs was first observed by Percy John Heawood **[5]** in 1898.

1.7 Theorem

A maximal planar graph G of order 3 or more has chromatic number 3 if and only if G is Eulerian.

Proof. Let there be given a planar embedding of G. suppose first that G is not Eulerian. Then G contains a vertex v of odd degree $k \geq 3$. Let

$$N(v) = \{v_1, v_2, \dots, v_k,\},$$

Where $C = (v_1, v_2 \dots v_k, v_1)$ is an odd cycle in G. Because v is adjacent to every vertex of C, it follows that $\chi(G) = 4$.

We verify the converse by induction on the order of maximal planar Eulerian graphs. If the order of G is 3, then $G = K_3$ and $\chi(G) = 3$. Assume that every maximal planar Eulerian graph of order k has chromatic number 3 for an integer $k \geq 3$ and let G be a maximal planar Eulerian graph of order $k + 1$. Let there be given a planar embedding of G and let uw be an edge of G. Then uw is on the boundary of two (triangular) regions of G. Let x be the third vertex on the boundary of one of these regions and y the third vertex on the boundary of the other region. Suppose that

$N(x) = \{u=x_1, x_2, \dots x_k = w\}$ and $N(y) = \{u= y_1, y_2, \dots y_l = w\}$,

Where k and ℓ are even, such that $C = (x_1, x_2, \dots x_k, x_1)$ and $C' = (y_1, y_2, \dots y_l, y_1)$ are even cycles. Let G' be the graph obtained from G by (1) deleting x, y, and uw from G and (2) adding a new vertex z and joining z to every vertex of C and C'. Then G' is a maximal planar Eulerian graph of order k. By the induction hypothesis, $\chi(G') = 3$. According to Theorem1.7, G' is uniquely colorable. Since z is adjacent to every vertex of C and C' we may assume that z is colored 1 and that the vertices of C and C' alternate in the colors 2 and 3. From the 3-coloring of G', a 3-coloring of G can be given where every vertex of $V(G) - \{x, y\}$ is assigned the same color as in G' and x and y are colored 1.

On the other hand, Chartrand and Geller **[1]** showed that every uniquely 4-colorable planar graph must be maximal planar.

2. Perfect Graphs

2.1 Definition

For any graph G, if $\chi(G) = \omega(G)$, then g is called perfect graph. While there are many examples of graphs G for which $\chi(G) = \omega(G)$, such as complete graphs and bipartite graphs, there are also many graphs whose chromatic number exceeds its clique number such as the Petersen graph and the odd cycles of length 5 or more. As we are about to see, the chromatic number of a graph can be considerably larger than its clique number. The fact that a graph can be triangle-free and yet have a large chromatic number has been established by a number of mathematicians, including Blanche Descartes **[2]** John Kelly and Leroy Kelly **[6],** and Alexander Zykov **[10],** Jan Mycielski **[9].**

2.2 Theorem

Every bipartite graph is perfect.

Proof. Let G be a bipartite graph and let H be an induced sub graph of G. If H is nonempty, then $\chi(H) = \omega(H) = 2$; while if H is empty, then $\chi(H) = \omega(H) = 1$. In either case, $\chi(H) = \omega(H)$ and so G is perfect.

2.3 The Perfect Graph Conjecture

A graph is perfect if and only if its complement is perfect. In 1972, Laszlo Lovasz **[7]** showed that this conjecture is, in fact, true.

2.4 Theorem

Every interval graph is perfect.

Proof. Let G be an interval graph with $V(G) = \{v_1, v_2 ..., v_n\}$. Since every induced sub graph of an interval graph is also an interval graph, it suffices to show that $\chi(G) = \omega(G)$. Because G is an interval graph, there exist n closed intervals $I_i = [a_i, b_i]$, $1 \le i \le n$, such that v_i is adjacent to v_j $(i \neq j)$ if and only if $I_i \cap I_j \neq \phi$. We may assume that the intervals (and consequently, the vertices of G) have been labeled so that $a_1 \le a_2 \le ... \le a_n$.

We now define a vertex coloring of G. First, assign v_1 the color 1. If v_1 and v_2 are not adjacent (that is, if I_1 and I_2 are disjoint), then assign v_2 the color 1 as well; otherwise, assign v_2 the color 2. Proceeding inductively, suppose that we have assigned colors to v_1, v_2, ... , v_r where $1 \le r < n$; We now assign v_{r+1} the smallest color (positive integer) that has not been assigned to any neighbor of v_{r+1} in the set $\{v_1, v_2, ... , v_r\}$. Thus if v_{r+1} is adjacent to no vertex in $\{v_1, v_2, ..., v_r\}$, then v_{r+1} is assigned the color 1. This gives a k-coloring of G for some positive integer k and so $\chi(G) \le k$. If $k = 1$, then $G = \overline{K}_n$ and $\chi(G) = \omega(G) = 1$. Hence we may assume that $k \ge 2$.

Suppose that the vertex v_t has been assigned the color k. Since it was not possible to assign v_t any of the colors 1,2,...k-1, this means that the interval $I_t = [a_t, b_t]$ must have a nonempty intersection with k-1 intervals $I_{j1}, I_{j2}..... , I_{jk-1}$ where say $1 \le j_1 < j_2 < ... < j_{k-1} < t$. Thus $a_{jl} \le a_{j2} \le ...., \le a_{jk-1} \le a_t$. Since $I_{ji} \cap I_t \neq \phi$ for $1 \le i \le k - 1$, it follows that

$$a_t \in I_{j1} \cap I_{j2} \cap .... \cap I_{jk-1} \cap I_t$$

Thus for $U = \{v_{j1}, v_{j2}, ..., v_{jk-1}, v_t\}$,

$$G[U] = K_k$$

and so $\chi(G) \le k \le \omega(G)$. Since $\chi(G) \ge \omega(G)$, we have $\chi(G) = \omega(G)$, as desired.

2.5 Note

We now consider a more general class of graphs. Recall that a chord of a cycle C in a graph is an edge that joins two non-consecutive vertices of C. For example, wz and xz are chords in the cycle $C = (u, v, w, x, y, z, u)$ in the graph G of Figure2; while in the cycle $C' = (w, x, y, z, w)$ in G, the edge xz is a chord and wz is not. The cycle $C'' = (u, v, w, z, u)$ has no chords. Obviously no triangle contains a chord.

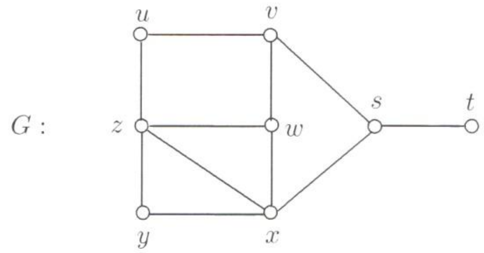

Figure (b). *Chords in cycles*

2.6 Definition

A graph G is a **chordal graph** if every cycle of length 4 or more in G has a chord. Since the cycle $C'' = (u, v, w, z, u)$ in the graph G of Figure2 contains no chords, the graph G is not a chordal graph.

While every complete graph is a chordal graph, no complete bipartite graph $K_{s,t}$, where $s,t \ge 2$, is chordal, for if u_1 and v_1 belong to one partite set and u_2 and v_2 belong to the other partite set, then the cycle $(u_1, u_2, v_1, v_2, u_1)$ contains no chord. Indeed, no graph having girth 4 or more is chordal. The graphs G_1 and G_2 of Figure 2.1 are chordal graphs. For the subset $S_1 = \{u_1, v_1, x_1\}$ of $V(G_1)$ and, the subset $S_2 = \{u_2, w_2, x_2\}$ of $V(G_2)$, let the graph G_3 be obtained by identifying the vertices in the complete sub graph $G_I[S_1]$ with the vertices in the complete sub graph $G_2[S_2]$, where, say, u_1 and u_2 are identified, v_1 and x_2 are identified, and x_1 and w_2 are identified. The graph G_3 shown in Figure 2.1 is also a chordal graph.

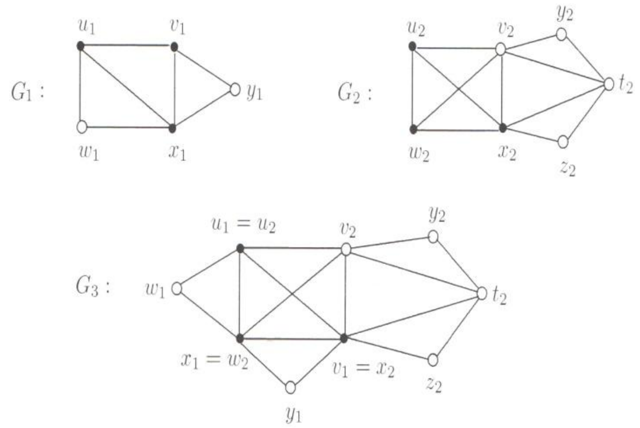

Figure 2.1. *Chordal graphs*

More generally, suppose that G_1 and G_2 are two graphs containing complete sub graphs H_1 and H_2, respectively, of the same order and G_3 is the graph obtained by identifying the vertices of H_1 with the vertices of H_2 (in a one-to-one manner). If G_3 contains a cycle of length 4 or more having no chord, then C must belong to G_1 or G_2. That is, if G_1 and G_2 are chordal, then G_3 is chordal. Furthermore, if G_3 is chordal, then both G_1 and G_2 are chordal.

We have now observed that every graph obtained by identifying two complete sub graphs of the same order in two chordal graphs is also chordal. These are not only sufficient conditions for a graph to be chordal. They are necessary conditions as well. The following characterization of chordal graphs is due to Andras Hajnal and Janos Suranyi **[4]** and Gabriel

Dirac **[3].**

2.7 Theorem

A graph G is chordal if and only if G can be obtained by identifying two complete sub graphs of the same order in two chordal graphs.

Proof. From our earlier observations, we need only show that every chordal graph can be obtained from two chordal graphs by identifying two complete sub graphs of the same order in these two graphs. If G is complete, say $G = K_n$, then G is chordal and can trivially be obtained by identifying the vertices of $G_1 = K_n$ and the vertices of $G_2 = K_n$ in any one-to-one manner. Hence we may assume that G is a connected chordal graph that is not complete.

Let S be a minimum vertex-cut, of G. Now let V_1 be the vertex set of one component of $G - S$ and let $V_2 = V(G) - (V_1 \cup S)$. Consider the two S-branches. $G_1 = G[V_1 \cup S]$ *and* $G_2 = G[V_2 \cup S]$of G. Consequently, G is obtained by identifying the vertices of S in G_1 and G_2. We now show that $G[S]$ is complete. Since this is certainly true if $|S| = 1$, we may assume that $|S| \geq 2$.

Each vertex v in S is adjacent to at least one vertex in each component of $G - S$, for otherwise $S- \{v\}$ is a vertex-cut of G, which is impossible. Let $u, w \in S$. Hence there are $u - w$ paths in G_1, where every vertex except u and w belongs to V_1. Among all such paths, let $P = (u, x_1, x_2 \ldots . x_s, w)$ be one of minimum length. Similarly, let $P' = (u, y_1, y_2 \ldots, y_t, w)$ be a $u - w$ path of minimum length where every vertex except u and w belongs to V_2.

Hence

$$C = (u, x_1, x_2 \ldots . x_S, w, y_t, y_{t-l}, \ldots y_1, u)$$

is a cycle of length 4 or more in G. Since G is chordal, C contains a chord. No vertex x_i $(1 \leq i \leq s)$ can be adjacent to a vertex y_j $(1 \leq j \leq t)$ since S is a vertex-cut of G.

Furthermore, no non-consecutive vertices of P or of P' can be adjacent due to the manner in which P and P' are defined. Thus $uw \in E(G)$, implying that $G[S]$ is complete. G_1 and G_2 are chordal.

2.8 Corollary

Every chordal graph is perfect.

Proof. Since every induced sub graph of a chordal graph is also a chordal graph, it suffices to show that if G is a connected chordal graph, then $\chi(G) = \omega(G)$. We proceed by induction on the order n of G. If $n = 1$, then $G = K_1$ and $\chi(G) = \omega(G) = 1$. Assume therefore that $\chi(H) = \omega(H)$ for every chordal graph H of order less than n, where $n \geq 2$ and let G be a chordal graph of order $n \geq 2$.

If G is a complete graph, then $\chi(G) = \omega(G) = n$. Hence we may assume that G is not complete. G can be obtained from two chordal graphs G_1 and G_2 by identifying two complete sub graphs of the same order in G_1 and G_2. Observe that

$$\chi(G) \leq \max\{x(G_l), \chi(G_2)\} = k.$$

By the induction hypothesis, $\chi(G_1) = \omega(G_1\}$ and $\chi(G_2) = \omega(G_2)$.

Thus $\chi(G) \leq \max\{\omega(G_1), \omega(G_2)\} = k$. On the other hand, let S denote the set of vertices in G that belong to G_1 and G_2. Thus $G[S]$ is complete and no vertex in $V(G_1) - S$ is adjacent to a vertex in $V(G_2) - S$. Hence $\omega(G) = \max\{\omega(G_1), \omega(G_2)\} = k$.

Thus $\chi(G) \geq k$.

Therefore, $\chi(G) = k = \omega(G)$.

2.9 Definition

We now consider a class of perfect graphs that can be obtained from a given perfect graph. Let G be a graph where $v \in V(G)$. Then the **replication graph** $R_v(G)$ of G (with respect to v) is that graph obtained from G by adding a new vertex v' to G and joining v' to the vertices in the closed neighborhood $N[v]$ of v. In 1972 Laszlo Lovasz [8] obtained few results.

2.10 Theorem

Let G be a graph where $v \in V(G)$. If G is perfect, then $R_v(G)$ is perfect.

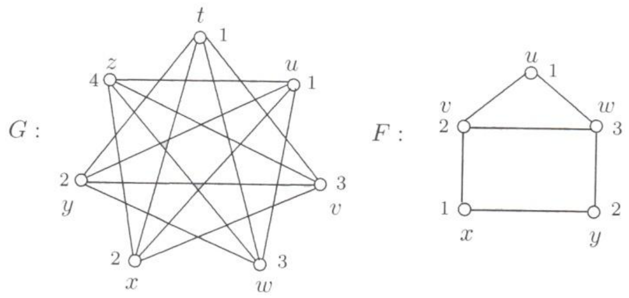

Figure 2.2. *Non-chordal graphs*

Proof. Let $G' = R_v(G)$. First, we show that $\chi(G') = \omega(G')$. We consider two cases, depending on whether v belongs to a maximum clique of G.

Case 1. V belongs to a maximum clique of G. Then $\omega(G') = \omega(G) + 1$. Since

$$\chi(G') \leq \chi(G) + 1 = \omega(G) + 1 = \omega(G'),$$

it follows that $\chi(G') = \omega(G')$.

Case 2. V does not belong to any maximum clique of G. Suppose that $\chi(G) = \omega(G) = k$. Let there be given a k-coloring of G using the colors $1, 2 \ldots k$. We may assume that v is assigned the color 1. Let V_1 be the color class consisting of the vertices of G that are colored 1. Thus $v \in V_l$. Since $\omega(G) = k$, every maximum clique of G must contain a vertex of each color. Since v does not belong to a maximum clique, it follows that $|V_1| \geq 2$. Let $U_1 = V_1 - \{v\}$. Because every maximum clique of G contains a vertex of U_1, it follows that $\omega(G\text{-}U_1) = \omega(G)\text{-}1 = k - 1$. Since G is perfect, $\chi(G–U_1) = k -1$. Let a $(k - 1)$-coloring of $G - U_1$ be given, using the colors $1, 2 \ldots k - 1$. Since V_l is an independent set of vertices, so is $U_1 \cup \{v^1\}$. Assigning the vertices of $U_1 \cup \{v'\}$ the color k produces a k-coloring of G'. Therefore,

$$k = \omega(G) \leq \omega(G') \leq \chi(G') \leq k$$

and so $\chi(G') = \omega(G')$.

It remains to show that $\chi(H) = \omega(H)$ for every induced sub graph H of G'.

This is certainly the case if H is a sub graph of G. If H contains v' but not v, then $H \cong G[(V(H)\text{-}\{v'\}) \cup \{v\}]$ and so $\chi(H) = \omega(H)$. If H contains both v and v' but $H \ncong G'$, then H is the replication graph of $G[V(H) - \{v'\}]$ and the argument used to show that $\chi(G') = \omega(G')$ can be applied to show that $\chi(H) = \omega(H)$.

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