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On some generalized results on $\psi - |\bar{N}, p_n, \delta, \gamma|_k$-summability factors of Infinite Series

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Abstract: *In this paper I have proved a theorem on* $\psi - |\bar{N}, p_n, \delta, \gamma|_k -$ *summability factors which generalizes some previous known results and gives some unknown result.*

Keywords: *Weighted mean Summability, Summability factors, Infinite series.*

AMS subject classification: *40D15, 40F05, 40G99.*

1. INTRODUCTION

Let (ψ_n) be sequence of positive real number, let Σa_n be a given infinite series with partial sums (s_n) and (t_n) denote the n-th Cesaro means of the sequence (na_n). Then the series Σa_n is said to be summable $|C,1|_k, k \geq 1$ if (Flett [3]).

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty \tag{1.1}$$

and it is said to be summable $\psi - |C,1|_k, k \geq 1$ if (Seyhan [6]).

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} |t_n|^k < \infty \tag{1.2}$$

If we are taking $\varphi_n = n, \psi - |C,1|_k$ -summability reduces to $|C,1|_k$ -summability.

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=1}^{n} p_v \to \infty \text{ as } n \to \infty \tag{1.3}$$

The sequence to sequence transformation

$$u_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v s_v \tag{1.4}$$

defines the sequence (u_n) of the $(\bar{N}, p_n)$ mean of the sequences (s_n) generated by the sequence of coefficients (p_n) (Hardy [4]).

The series Σa_n is said to be summable $|\bar{N}, p_n|_k, k \geq 1$ if (Bor [1])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|\Delta u_{n-1}|^k<\infty \tag{1.5}$$

and it is said to summable $|\bar{N},p_n,\delta|_k, k\geq 1$ an d $\delta\geq 0$ if (Bor [2])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|\Delta u_{n-1}|^k<\infty \tag{1.6}$$

where $\Delta u_{n-1}=u_n-u_{n-1}=\frac{-p_n}{P_nP_{n-1}}\sum_{v=1}^{n}p_{v-1}a_v : n\geq 1$

and Σa_n is said to summable $|\bar{N},p_n,\delta,\gamma|_k, k\geq 1,\delta\geq 0$ and $\gamma\geq 1$ if

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)}|\Delta u_{n-1}|^k<\infty \tag{1.7}$$

and it is said to summable $\psi-|\bar{N},p_n,\delta,\gamma|_k, k\geq 1,\delta\geq 0,\gamma\geq 1$

$$\sum_{n=1}^{\infty}(\varphi_n)^{\gamma(\delta k+k-1)}|\Delta u_{n-1}|^k<\infty \tag{1.8}$$

If $\varphi_n=\frac{P_n}{p_n}$ then $\psi-|\bar{N},p_n\delta,\gamma|_k$-summability reduces to $|\bar{N},p_n,\delta,\gamma|_k$-summability and if $\varphi_n=n\delta=O$ and $\gamma=1$ then $\varphi-|\bar{N},p_n,\delta,\gamma|_k$-summability reduces to $|C,1|_k$-summability.

2. Known Results

Concerning $|C,1|_k$-summability, Mazhar [5] has proved the following theorem.

Theorem 2.1

If d $\lambda_m=O(1)$, as $m\to\infty$ (2.1)

$$\sum_{n=1}^{m}n\log n|\Delta^2\lambda_n|=O(1)\text{, as } m\to\infty \tag{2.2}$$

$$\sum_{v=1}^{m}\frac{|t_v|^k}{v}=O(\log m)\text{ as } m\to\infty \tag{2.3}$$

then the series $\Sigma a_n\lambda_n$ is summable $|C,1|_k, k\geq 1$.

And Sulaiman (7) has proved the following theorem.

Theorem 2.2 Let (φ_n) and (X_n) be sequences of positive real numbers such that (X_n) is non decreasing and condition (2.1) is satisfied

If $np_n=O(P_n),P_n=O(np_n)$, as $n\to\infty$ (2.4)

$$\beta_{n+1}=O(\beta_n) \tag{2.5}$$

$$\Delta\beta_n=O(n^{-1}\beta_n)\text{ as } n\to\infty \tag{2.6}$$

$$\sum_{n=1}^{\infty}nX_n|\Delta^2\lambda_n|=O(1) \tag{2.7}$$

$$\sum_{n=1}^{m} \frac{\varphi_n^{k-1} |\beta_v|^k |s_n|^k}{n^k X_n^{k-1}} = O(X_m) \text{ as } m \to \infty \tag{2.8}$$

$$\sum_{n=v}^{m} \frac{\varphi_n^{k-1}}{v^k P_{n-1}} = O\left(\frac{\varphi_v^{k-1}}{v^{k-1} P_v}\right)$$

then the series $\Sigma a_n \lambda_n \beta_n$ is summable $\varphi - |\bar{N}, p_n|_k, k \geq 1$.

3. MAIN RESULTS

The aim of this paper is to generalize the theorem (2.2), here I have proved the following theorem.

Theorem 3.1 Let (φ_n) and (X_n) be sequences of positive real numbers such that (X_n) is non decreasing and if the conditions (2.1), (2.4), (2.5), (2.6), (2.7) are satisfied.

$$\sum_{n=1}^{m} \frac{(\varphi_n)^{\gamma(k+\delta k-1)} |\beta_v|^k |s_n|^k}{n^k X_n^{k-1}} = O(X_m) \text{ as } m \to \infty \tag{3.1}$$

$$\sum_{n=v}^{m} \frac{(\varphi_n)^{\gamma(k+\delta k-1)}}{v^k P_{n-1}} = O\left(\frac{\varphi_v^{\gamma(\delta k+k-1)}}{v^{k-1} P_v}\right) \tag{3.2}$$

Then the series $\Sigma a_n \lambda_n \beta_n$ is summable $\varphi - |\bar{N}, p_n, \delta, \gamma|_k, k \geq 1, \gamma \geq 1$ and $\delta \geq 0$.

4. LEMMA

To prove the above theorem following Lemma is required.

Lemma 4.1 Sulaiman [7] The conditions (2.1) and (2.7) implies.

$$\sum_{n=1}^{\infty} X_n |\Delta\lambda_n| = O(1) \tag{4.1}$$

$$nX_n |\Delta\lambda_n| = \mathrm{O}(1) \text{ as } n \to \infty \tag{4.2}$$

$$X_n |\lambda_n| = O(1) \text{ as } n \to \infty \tag{4.3}$$

5. PROOF OF THE THEOREM 3.1

Let T_n be the $(\bar{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} a_n \lambda_n \beta_n$, we have

$$T_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v \sum_{r=1}^{v} a_r \lambda_r \beta_r$$
$$= \frac{1}{P_n} \sum_{v=1}^{n} (P_v - P_{v-1}) a_v \lambda_v \beta_v$$

And hence

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v \beta_v, n \geq 1.$$

Using Abeles transformation, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v \Delta(P_{v-1}\lambda_v \beta_v) + \frac{p_n}{P_n} \lambda_n \beta_n \mathrm{s}_n$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} (-p_v \lambda_v \beta_v s_v + P_v \lambda_v \Delta\beta_v s_v + P_v \beta_{v+1} \Delta\lambda_v s_v)$$

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}$$

Since $|T_{n1} + T_{n2} + T_{n3} + T_{n4}|^4 \le 4^k (|T_{n1}|^k + |T_{n2}|^k + |T_{n3}|^k + |T_{n4}|^k)$

In order to complete the proof, it is sufficient to show that

$$\sum_{n=1}^{\infty} (\varphi_n)^{\gamma(k+\delta k-1)} |T_{n,r}|^k < \infty, \;\; r = 1,2,3,4$$

Applying Hölders inequality, we have

$$\sum_{n=2}^{m} (\varphi_n)^{\gamma(\delta k+k-1)} |T_{n,1}|^k = \sum_{n=2}^{m} (\varphi_n)^{\gamma(\delta k+k-1)} \left| \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v \lambda_v \beta_v s_v \right|^k$$

$$= O(1) \sum_{n=2}^{m} \frac{(\varphi_n)^{\gamma(\delta k+k-1)} p_n^k}{P_n^k P_{n-1}^k} \left(\sum_{v=1}^{n-1} p_v |\lambda_v| |\beta_v| |s_v| \right)^k$$

$$= O(1) \sum_{n=2}^{m} \frac{(\varphi_n)^{\gamma(\delta k+k-1)} p_n^k}{P_n^k P_{n-1}} \sum_{v=1}^{n-1} p_v |\lambda_v|^k |\beta_v|^k |s_v|^k \left(\sum_{v=1}^{n-1} \frac{p_v}{P_{n-1}} \right)^{k-1}$$

$$= O(1) \sum_{v=1}^{m} p_v |\lambda_v|^k |\beta_v|^k |s_v|^k \sum_{n=v}^{m} \frac{(\varphi_n)^{\gamma(\delta k+k-1)} p_n^k}{P_n^k P_{n-1}^k}$$

$$= O(1) \sum_{v=1}^{m} p_v |\lambda_v|^k |\beta_v|^k |s_v|^k \sum_{n=v}^{m} \frac{(\varphi_n)^{\gamma(\delta k+k-1)}}{n^k P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} \frac{p_v (\varphi_v)^{\gamma(\delta k+k-1)}}{v^{k-1} p_v} |\lambda_v|^k |\beta_v|^k |s_v|^k$$

$$= O(1) \sum_{v=1}^{m} \frac{(\varphi_v)^{\delta k+k-1}}{v^k} |\lambda_v|^{k-1} |\beta_v|^k |s_v|^k |\lambda_v|$$

$$= O(1) \sum_{v=1}^{m} \frac{(\varphi_v)^{\gamma(\delta k+k-1)}}{v^k X_v^{k-1}} |\beta_v|^k |s_v|^k \sum_{n=v}^{\infty} \Delta |\lambda_n|$$

$$= O(1) \sum_{v=1}^{m} \frac{(\varphi_v)^{\gamma(\delta k+k-1)}}{v^k X_v^{k-1}} |\beta_v|^k |s_v|^k \sum_{n=v}^{\infty} |\Delta\lambda_n|$$

$$= O(1) \sum_{n=1}^{\infty} |\Delta\lambda_n| \sum_{v=1}^{n} \frac{(\varphi_v)^{\gamma(\delta k+k-1)}}{v^k X_v^{k-1}} |\beta_v|^k |s_v|^k$$

$$= O(1) \sum_{n=1}^{\infty} X_n |\Delta\lambda_n|$$

$$= O(1).$$

$$\sum_{n=2}^{m}(\varphi_n)^{\gamma(\delta k+k-1)}|T_{n,2}|^k=\sum_{n=2}^{m}(\varphi_n)^{\gamma(\delta k+k-1)}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_v\lambda_v\Delta\beta_v s_v\right|^k$$

$$=O(1)\sum_{n=1}^{m}\frac{(\varphi_n)^{\gamma(\delta k+k-1)}p_n^k}{P_n^kP_{n-1}^k}\left(\sum_{v=1}^{n-1}p_v|\lambda_v||\Delta\beta_v||s_v|\right)^k$$

$$=O(1)\sum_{n=1}^{m}\frac{(\varphi_n)^{\gamma(\delta k+k-1)}p_n^k}{P_n^kP_{n-1}^k}\left(\sum_{v=1}^{n-1}v^{-1}p_v|\lambda_v||\beta_v||s_v|\right)^k$$

$$=O(1)\sum_{n=1}^{m}\frac{\varphi_n^{\gamma(k+\delta k-1)}p_n^k}{P_n^kP_{n-1}}\sum_{v=1}^{n-1}v^{-1}p_v\left(\frac{P_v}{p_v}\right)^k|\lambda_v|^k|\beta_v|^k|s_v|^k\left(\sum_{v=1}^{n-1}\frac{p_v}{P_{n-1}}\right)^{k-1}$$

$$=O(1)\sum_{v=1}^{m}p_v|\lambda_v|^k|\beta_v|^k|s_v|^k\sum_{n=v}^{m}\frac{(\varphi_n)^{\gamma(\delta k+k-1)}p_n^k}{P_n^kP_{n-1}}$$

$$=O(1)\sum_{v=1}^{m}p_v|\lambda_v|^k|\beta_v|^k|s_v|^k\sum_{n=v}^{m}\frac{(\varphi_n)^{\gamma(\delta k+k-1)}}{v^kP_{n-1}}$$

$$=O(1)\sum_{v=1}^{m}\frac{(\varphi_v)^{\gamma(\delta k+k-1)}p_v}{v^{k-1}P_v}|\lambda_v|^k|\beta_v|^k|s_v|^k$$

$$=O(1)\sum_{v=1}^{m}\frac{(\varphi_v)^{\delta k+k-1}}{v^k}|\lambda_v|^k|\beta_v|^k|s_v|^k$$

$$=O(1),\text{ as in the case of }T_{n,1.}$$

$$\sum_{n=2}^{m}\varphi_n^{k-1}|T_{n,2}|^k=O(1)\sum_{n=2}^{m}(\varphi_n)^{\gamma(\delta k+k-1)}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_v\beta_{v+1}\Delta\lambda_v s_v\right|^k$$

$$=O(1)\sum_{n=1}^{m+1}\frac{(\varphi_n)^{\gamma(\delta k+k-1)}p_n^k}{P_n^kP_{n-1}^k}\left(\sum_{v=1}^{n-1}p_v|\beta_v||\Delta\lambda_v||s_v|\right)^k$$

$$=O(1)\sum_{n=1}^{m}\frac{(\varphi_n)^{\gamma(\delta k+k-1)}p_n^k}{P_n^kP_{n-1}^k}\sum_{v=1}^{n-1}\frac{P_v^k}{X_v^{k-1}}|\beta_v|^k|\Delta\lambda_v||s_v|^k\left(\sum_{v=1}^{n-1}X_v|\Delta\lambda_v|\right)^{k-1}$$

$$=O(1)\sum_{n=1}^{m}\frac{P_v^k}{X_v^{k-1}}|\beta_v|^k|\Delta\lambda_v||s_v|^k\sum_{n=v+1}^{m+1}\frac{(\varphi_n)^{\gamma(\delta k+k-1)}p_n^k}{v^kP_{n-1}^k}$$

$$=O(1)\sum_{v=1}^{m}\frac{P_v^k}{X_v^{k-1}}|\beta_v|^k|\Delta\lambda_v||s_v|^k\sum_{n=v+1}^{m+1}\frac{(\varphi_n)^{\gamma(\delta k+k-1)}}{v^kP_{n-1}}$$

$$=O(1)\sum_{v=1}^{m}\frac{P_v}{X_v^{k-1}}|\beta_v|^k|\Delta\lambda_v||s_v|^k\sum_{n=v+1}^{m+1}\frac{(\varphi_n)^{\gamma(\delta k+k-1)}}{v^kP_{n-1}}$$

$$=O(1)\sum_{v=1}^{m}\frac{(\varphi_v)^{\gamma(\delta k+k-1)}}{v^{k-1}X_v^{k-1}}|\beta_v|^k|\Delta\lambda_v||s_v|^k$$

$$=O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{(\varphi_v)^{\delta k+k-1}|\beta_v|^k|s_v|^k}{v^kX_v^{k-1}}$$

$$=O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v}\frac{(\varphi_v)^{\gamma(k+\delta k-1)}|\beta_r|^k|s_r|^k}{r^kX_r^{k-1}}$$

$$+O(1)m\,|\Delta\lambda_m|\sum_{v=1}^{m}\frac{(\varphi_v)^{\gamma(k+\delta k-1)}|\beta_v|^k|s_v|^k}{v^k X_v^{k-1}}$$

$$=O(1)\sum_{v=1}^{m-1}v\,|\Delta^2\lambda_v|\,X_v+O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|\,X_v+O(1)m\,|\Delta\lambda_m|\,X_m$$

$$=O(1)$$

$$\sum_{n=1}^{m}(\varphi_n)^{\gamma(k+\delta k-1)}|T_{n,4}|^k=\sum_{n=1}^{m}(\varphi_n)^{\gamma(\delta k+k-1)}\left|\frac{p_n}{P_nP_{n-1}}\beta_n\lambda_n s_n\right|^k$$

$$=O(1)\sum_{n=1}^{m}(\varphi_n)^{\gamma(\delta k+k-1)}\left(\frac{p_n}{P_n}\right)^k|\beta_n|^k|s_n|^k|\lambda_n|^{k-1}|\lambda_n|$$

$$=O(1)\sum_{n=1}^{m}\frac{(\varphi_n)^{\gamma(\delta k+k-1)}|\beta_n|^k|s_n|^k}{n^k X_n^{k-1}}\sum_{v=n}^{\infty}|\Delta\lambda_v|$$

$$=O(1)\sum_{v=1}^{m}|\Delta\lambda_n|\sum_{n=1}^{v}\frac{(\varphi_n)^{\gamma(\delta k+k-1)}|\beta_n|^k|s_n|^k}{n^k X_n^{k-1}}$$

$$=O(1)\sum_{v=1}^{m}X_v\,|\Delta\lambda_v|$$

$$=O(1)$$

This completes the proof of the theorem.

6. Corollary

This theorem have the following results as corollaries.

Corollary 6.1

If we are taking $\psi=\frac{P_n}{p_n}$ then the infinite series $\Sigma a_n\lambda_n\beta_n$ is $|\bar{N},p_n,\delta,\gamma|_k$-summable $\delta\geq 0,\gamma\geq 1$ and $k\geq 1$.

Corollary 6.2

If we are taking $\delta=0,\gamma=1$ then the infinite series $\Sigma a_n\lambda_n\beta_n$ is $\psi-|\bar{N},p_n|_k$-summable, $k\geq 1$.

Corollary 6.3

If we are taking $\delta=0,\gamma=1,\psi=\frac{P_n}{p_n}$ then the infinite series $\Sigma a_n\lambda_n\beta_n$ is $|\bar{N},p_n|_k$-summable $k\geq 1$.

Corollary 6.4

If we are taking $\varphi=n$ then the infinite series. $\Sigma a_n\lambda_n\beta_n$ is $|C,1,\delta,\gamma|_k$-summable $\delta\geq 0,\gamma=1$ and $k\geq 1$.

Corollary 6.5

If we are taking $\varphi=n$, $\delta=0,\gamma=1$ then the infinite series $\Sigma a_n\lambda_n\beta_n$ is $|C,1|_k$-summable, $k\geq 1$.

7. Conclusion

The results of this theorem is more general rather than the results of any other previous proved theorem, which will be enrich the literate of summability theory of infinite series.

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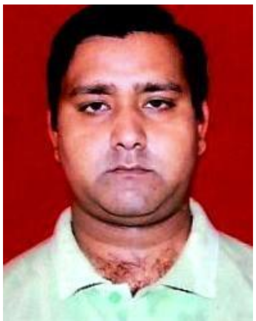

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