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Steady Flow of a Viscous Incompressible Fluid through Long Tubes Employing a Complex Variable Technique

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Abstract: *A steady flow of a viscous fluid through straight long non-circular tubes under a pressure gradient down the tube length is discussed in the paper.*

The fluid is assumed to be homogenous and incompressible. For such a flow the pressure gradient has to be a constant. The axial velocity of the flow satisfies Poisson's equation together with the no-slip condition on the boundary Γ *of the tube.*

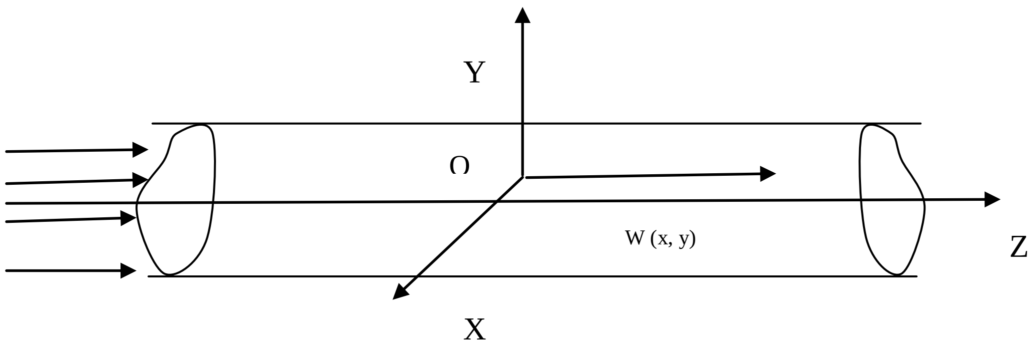
A complex variable technique has been developed to solve the problem. This technique can be successfully applied whenever the tube cross section Γ *can be expressed in the form* $Z\bar{Z} = \phi\ Z + \overline{\phi\ Z}$ *where* $Z = x + iy$ *and* $\bar{Z} = x - iy$ *it's conjugate. Farther* $\phi\ Z$ *is a regular function and* $\overline{\phi\ Z} = \bar{\phi}\ \bar{Z}$ *is it conjugate.*

This technique is employed to find the velocity field for the following tube-cross sections is. 1. Ellipse, 2. Equilateral triangle, 3.Hyperbola bounded by a chord, 4. Two co-axial hyperbolas, 5. Two conjugate hyperbolas and 6. Lune bounded by two circles.

1. Introduction

Consider a viscous incompressible homogenous fluid flowing steadily through a long straight horizontal tube under a pressure gradient. Such a flow through a circular tube was considered earlier in the year 1846 by J.L.M. Poisceuille a French physician in connection with the estimation of the flow rate of blood through arteries'. This has been treated by several authors of books classical test books on fluid mechanics.

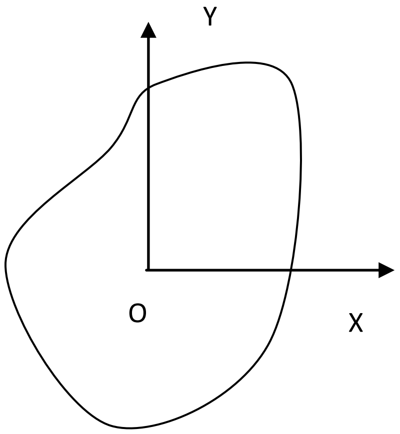
Consider a Cartesian System O (x, y, z) with the co -axis along the tube length and x and y axes in a plane perpendicular to tube length with the origin 'O' can be taken anywhere on the axis.



The flow is under a pressure gradient down the tube length. Let fluid velocity be taken

$$\bar{q} = (o, o, w) \tag{1}$$

The velocity satisfies the equation of the continuity



Div $\vec{q} = 0 \Rightarrow \frac{\partial w}{\partial z} = 0$ (2)

$\therefore$ W is independent of Z

$\Rightarrow$ W is function of x and y and t

Since the flow is steady, W is independent of time’t’ W is a function of x and y only

The momentum equation (Navier-stokes’ equation) that characterized the steady flow in the absent of the external force

$$\rho \frac{d\bar{q}}{dx} = -\nabla p + \mu D^2 \bar{q} \quad (3)$$

Where $\vec{q}$ = fluid velocity, ρ = fluid density (assumed constant), P = pressure, μ = viscosity coefficient

Equation of motion in the x-direction

$$\frac{\partial p}{\partial x} = 0 \quad (4)$$

Equation of motion in the y-direction

$$\frac{\partial p}{\partial y} = 0 \quad (5)$$

Equation of motion in the z –direction

$$0 = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \quad (6)$$

Differentiating with respect to ‘z ’we get

$$0 = -\frac{\partial^2 p}{\partial z^2} + \mu\left[\frac{d}{dz}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\right]$$

$$0 = -\frac{\partial^2 p}{\partial z^2} + \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{dw}{dz}$$

$\therefore \frac{\partial^2 p}{\partial z^2} = 0 \left(\because \frac{\partial w}{\partial z} = 0\right) \Rightarrow \frac{\partial p}{\partial z}$ is independent of ‘z’ (7)

From the equations (4), (5) and (7)

$\therefore \frac{\partial p}{\partial z}$ = C (constant)

The equation (6) that satisfied by the axial component w(x,y) of the velocity is give by the Poisson equation.

$$\nabla^2 w = c\mu = constat \tag{8}$$

with the boundary condition,

W=0 on the boundary Γ on the tube (9)

Proposed Complex variable technique for solving the equation (8) for the velocity fluid(w)

Let us Introduce the complex variables Z and $\overline{Z}$

$$Z = x + iy \text{ and } \overline{Z} = x - iy \tag{10}$$

The equations can now to transformation as $\int \frac{\partial^2 p}{\partial z \partial \overline{Z}} = \frac{c}{4\mu}$ (11)

Integrating (11) with Z

$$\frac{\partial p}{\partial z} = \frac{c}{4\mu} Z + fuction \tag{12}$$

This on integrating with repeat to $\overline{Z}$ yield ds

$$w = -\frac{c}{4\mu} Z\overline{Z} + function + fuctions\overline{Z} \tag{13}$$

This equation of W is a real function of X and Y of Z and $\overline{Z}$ and on the R.H.S. $Z\overline{Z}$ is real.

Therefore The Velocity w can be expressed as

$$w = -\frac{c}{4\mu} Z\overline{Z} + f(Z) + \overline{f(Z)} \tag{14}$$

where f(z) is a regular function of Z and $\overline{f(Z)} = \overline{f}(\overline{Z})$ is conjugate function of f(Z)

The function f (z) is to be so choose such that the no-slip condition on the $w = 0$ on the boundary Γ

Therefore on the boundary Γ of the tube

$$f(Z) + \overline{f(Z)} = -\frac{c}{4\mu} Z\overline{Z} \tag{15}$$

This method is successful whenever the equation to the boundary Γ can be expressed as

$$Z\overline{Z} = \phi(Z) + \overline{\phi(Z)} \tag{16}$$

Expressed the equation we noticed that

$$f(Z) + \overline{f(Z)} = -\frac{c}{4\mu}\left[\phi(Z) + \overline{\phi(Z)}\right] \tag{17}$$

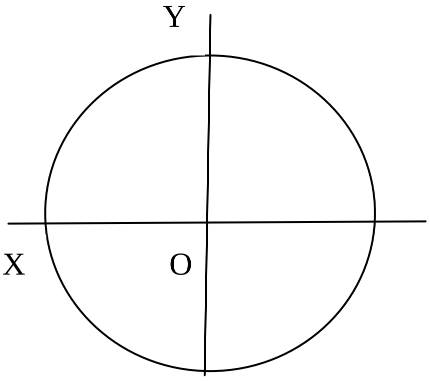
From (17) it can be noticed from the equation that (17)

$$f(Z) = -\frac{c}{4\mu}\phi(Z) \text{ and } \overline{f(Z)} = -\frac{c}{4\mu}\overline{\phi(Z)} \tag{18}$$

$\therefore$ The velocity in this given be the equation (14)

2. APPLICATIONS

A) Cross section bounded circular tube:



$$\Gamma : x^2 + y^2 - a^2 = 0 \tag{A.1}$$

$$\Gamma : Z\overline{Z} = a^2 \tag{A.2}$$

Here $Z\overline{Z} = \frac{a^2}{2} + \frac{a^2}{2} = \phi(Z) + \overline{\phi(Z)}$

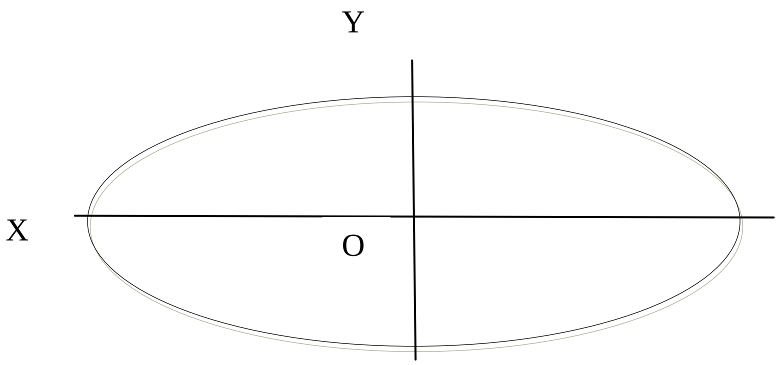
$$\therefore \phi(Z) = \frac{a^2}{2} = \overline{\phi(Z)} \tag{A.3}$$

$$\overline{f(Z)} = -c\frac{a^2}{8\mu} = f(Z) \tag{A.4}$$

$$w = -\frac{c}{4\mu} Z\overline{Z} + f(Z) + \overline{f(Z)}$$

$$-\frac{c}{4\mu}\left(Z\overline{Z} - a^2\right) = -\frac{c}{4\mu}\left(a^2 - x^2 - y^2\right) \tag{A.5}$$

B) Cross section bounded Ellipse:



$$\Gamma : \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{B.1}$$

This can be expressed as

$$b^2x^2 + a^2y^2 - a^2b^2 = 0$$

$$b^2\left(\frac{Z+\overline{Z}}{2}\right)^2 + a^2\left(\frac{Z-\overline{Z}}{2i}\right)^2 - a^2b^2 = 0$$

This can be written as

$$Z\overline{Z} = \frac{a^2-b^2}{4\ a^2-b^2}\ \ Z^2-\overline{Z}^2\ \ + \frac{a^2b^2}{2\ a^2+b^2}$$

$$= \frac{1}{4\ a^2-b^2}\left[\ a^2-b^2\ \ Z^2 + a^2b^2\right] + \left[\ a^2-b^2\ \ \overline{Z}^2 + a^2b^2\right] \qquad \text{(B.2)}$$

This is a second degree in Z and $\overline{Z}$

$$\phi\ \ Z\ = \frac{1}{4\ a^2-b^2}\left[\ a^2-b^2\ \ Z^2 + a^2b^2\right] \qquad \text{(B.3)}$$

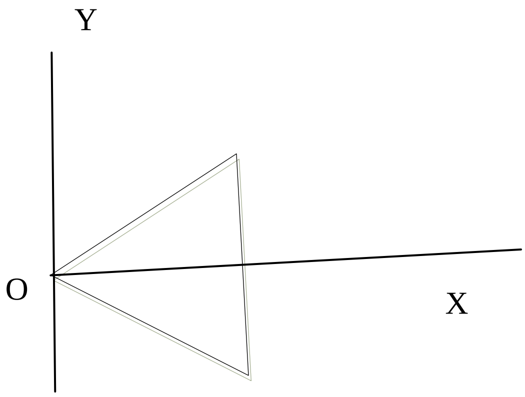
$$\frac{1}{4(a^2+b^2)}\overline{\phi\ \ Z}\ = \left[\ a^2-b^2\ \ \overline{Z}^2 + a^2b^2\right] \qquad \text{(B.4)}$$

$$f\ \ Z\ = -\frac{c}{4\mu}\phi\ \ Z \quad \text{and} \quad \overline{f\ \ Z}\ = -\frac{c}{4\mu}\overline{\phi\ \ Z}$$

The velocity is the equation substituting we get

$$w = -\frac{c}{4\mu}Z\overline{Z} + f\ \ Z\ + \overline{f\ \ Z}\ = -\frac{c}{4\mu}\ \ b^2x^2 + a^2y^2 - a^2b^2 \qquad \text{(B.5)}$$

C) Cross section bounded Equilateral triangle tube :



$$\Gamma:\ x-a\quad x+\sqrt{3}y\quad y-\sqrt{3}x\ = 0 \qquad \text{(C.1)}$$

This can be written as $x-a\quad x^2-3y^2\ = 0$

$$Z\overline{Z} = = \frac{1}{2a}\left[\ -Z^3 + 2aZ^2\ \ +\ -\overline{Z}^3 + 2a\overline{Z}^2\ \right] \qquad \text{(C.2)}$$

This is a cubic in Z and $\overline{Z}$

$$\phi(Z) = \frac{1}{2a}\left[-Z^3 + 2aZ^2 \right] \quad \text{(C.3)}$$

$$\overline{\phi(Z)} = \frac{1}{2a}\left[-\overline{Z}^3 + 2a\overline{Z}^2 \right] \quad \text{(C.4)}$$

$$f(Z) = -\frac{c}{4\mu}\phi(Z) \quad \text{and} \quad \overline{f(Z)} = -\frac{c}{4\mu}\overline{\phi(Z)} \quad \text{(C.5)}$$

$$\Rightarrow f(Z) = -\frac{c}{8\mu}\left[-Z^3 + 2aZ^2 \right] \quad \text{(C.6)}$$

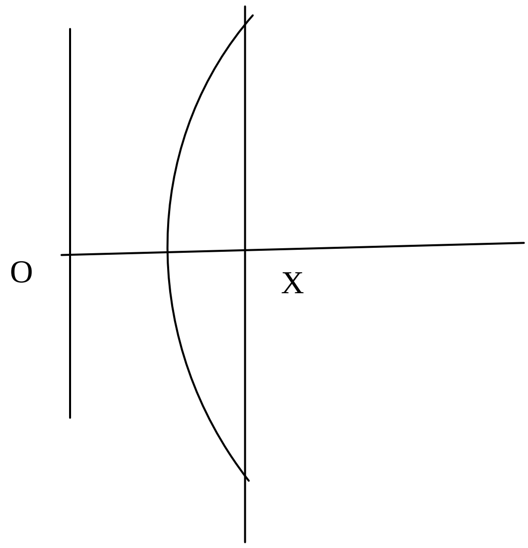
$$\overline{f(Z)} = -\frac{c}{8\mu}\left[-\overline{Z}^3 + 2a\overline{Z}^2 \right] \quad \text{(C.7)}$$

The velocity is now given by $w = -\frac{c}{4\mu} Z\overline{Z} + f(Z) + \overline{f(Z)}$

$$w = -\frac{c}{4\mu} Z\overline{Z} - \frac{c}{8a\mu}\left[-Z^3 + 2aZ^2 \right] - \frac{c}{8a\mu}\left[-\overline{Z}^3 + 2a\overline{Z}^2 \right]$$

$$\Rightarrow w = \frac{c}{4\mu}\left[(x-a)(x^2 - 3y^2) \right] \quad \text{(C.8)}$$

D) Cross section hyperbola bounded by a chord:



$$\Gamma : (x-a)(x^2 - 3y^2 - b^2) = 0 \text{ we get } Z\overline{Z} \quad \text{(D.1)}$$

$$Z\overline{Z} = \frac{1}{2a}\left[\left(-Z^3 + 2bZ^2 + a^2 Z + a^2 b\right) + \left(-\overline{Z}^3 + 2a\overline{Z}^2 + a^2\overline{Z} + a^2 b\right) \right] \quad \text{(D.2)}$$

This a cubic in Z and $\overline{Z}$

$$\phi(Z) = \frac{1}{2a}\left[-Z^3 + 2bZ^2 + a^2 Z + a^2 b \right] \tag{D.3}$$

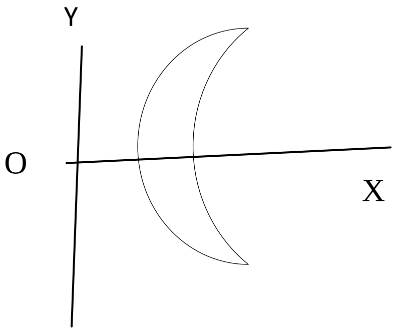
$$\overline{\phi(Z)} = \frac{1}{2a}\left[-\overline{Z}^3 + 2a\overline{Z}^2 + a^2\overline{Z} + a^2 b \right] \tag{D.4}$$

$$f(Z) = -\frac{c}{4\mu}\phi(Z) \quad \text{and} \quad \overline{f(Z)} = -\frac{c}{4\mu}\overline{\phi(Z)} \tag{D.5}$$

Values substituting get $w = -\frac{c}{4\mu}Z\overline{Z} + f(Z) + \overline{f(Z)} \Rightarrow w = \frac{c}{4\mu}\left[(x-a)(x^2 - 3y^2 - b^2) \right]$

(D.6)

E) Cross section bounded by two coaxial hyperbolas



$$\Gamma : \left(x^2 - \frac{y^2}{3-2\sqrt{2}} - a^2 \right)\left(x^2 - \frac{y^2}{3+2\sqrt{2}} - b^2 \right) = 0 \tag{E.1}$$

On simplifying we get $Z\overline{Z}$

$$Z\overline{Z} = \frac{-1}{2\left((\sqrt{2}+1)b^2 - (\sqrt{2}-1)a^2\right)}\left(Z^4 - \left[b^2(2+\sqrt{2}) - a^2(2-\sqrt{2})Z^2 + a^2b^2\right] + \overline{Z}^4 + \left[b^2(2+\sqrt{2}) - a^2(2-\sqrt{2})\overline{Z}^2 + a^2b^2\right]\right)$$

(E.2)

$$\phi(Z) = \frac{Z^4 - \left[b^2(2+\sqrt{2}) - a^2(2-\sqrt{2})Z^2 + a^2b^2\right]}{2\left((\sqrt{2}+1)b^2 - (\sqrt{2}-1)a^2\right)} \tag{E.3}$$

$$\overline{\phi(Z)} = \frac{\overline{Z}^4 + \left[b^2(2+\sqrt{2}) - a^2(2-\sqrt{2})\overline{Z}^2 + a^2b^2\right]}{2\left((\sqrt{2}+1)b^2 - (\sqrt{2}-1)a^2\right)} \tag{E.4}$$

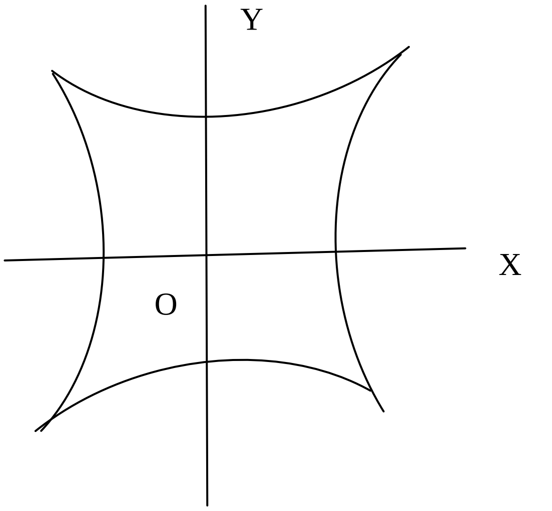
$$f(Z) = -\frac{c}{4\mu}\phi(Z) \quad \text{and} \quad \overline{f(Z)} = -\frac{c}{4\mu}\overline{\phi(Z)} \tag{E.5}$$

A fourth degree expression in Z and $\overline{Z}$

$$w = -\frac{c}{4\mu}Z\overline{Z} + f(Z) + \overline{f(Z)}$$

$$w = -\frac{c}{4\mu}\left[(3-2\sqrt{2})x^2 - y^2 - a^2(3-2\sqrt{2}) \right]\left[(3+2\sqrt{2}) \right] x^2 - y^2 - b^2(3+2\sqrt{2}) \tag{E.6}$$

F) Cross section bounded by a hyperbola and its conjugate:



$$\Gamma:\left(x^2-\frac{y^2}{3-2\sqrt{2}}+a^2\right)\left(x^2-\frac{y^2}{3+2\sqrt{2}}-b^2\right)=0 \qquad \text{(F.1)}$$

$$Z\bar{Z}=\frac{-1}{2\left(\sqrt{2}+1\right)b^2-\left(\sqrt{2}-1\right)a^2}\left(Z^4+\left[b^2\left(2+\sqrt{2}\right)-a^2\left(2-\sqrt{2}\right)Z^2+a^2b^2\right]+\bar{Z}^4+\left[b^2\left(2+\sqrt{2}\right)-a^2\left(2-\sqrt{2}\right)\bar{Z}^2+a^2b^2\right]\right)$$

(F.2)

This is a fourth degree expression in Z and $\bar{Z}$

$$\phi(Z)=\frac{Z^4-\left[b^2\left(2+\sqrt{2}\right)-a^2\left(2-\sqrt{2}\right)Z^2-a^2b^2\right]}{2\left(\sqrt{2}+1\right)b^2-\left(\sqrt{2}-1\right)a^2} \qquad \text{(F.3)}$$

$$\overline{\phi(Z)}=\frac{\bar{Z}^4+\left[b^2\left(2+\sqrt{2}\right)-a^2\left(2-\sqrt{2}\right)\bar{Z}^2+a^2b^2\right]}{2\left(\sqrt{2}+1\right)b^2-\left(\sqrt{2}-1\right)a^2} \qquad \text{(F.4)}$$

$$f(Z)=-\frac{c}{4\mu}\phi(Z) \quad \text{and} \quad \overline{f(Z)}=-\frac{c}{4\mu}\overline{\phi(Z)} \qquad \text{(F.5)}$$

Hence the velocity is give by $w=-\frac{c}{4\mu}Z\bar{Z}+f(Z)+\overline{f(Z)}$

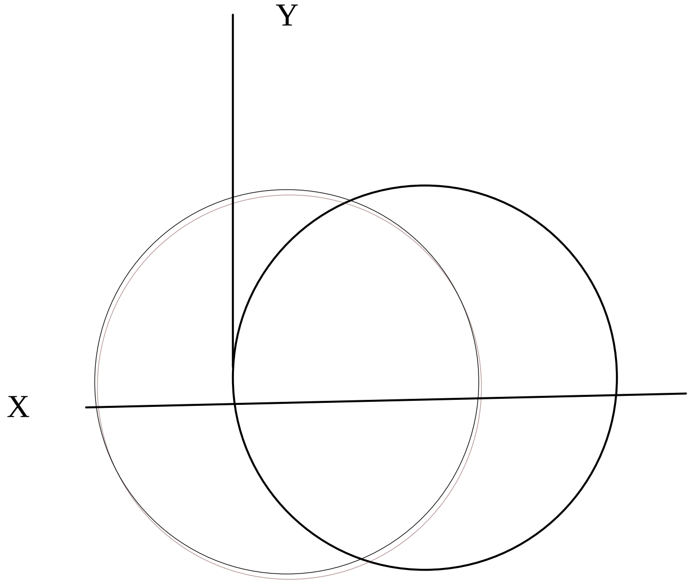
$$w=-\frac{c}{4\mu}\left[\left(3-2\sqrt{2}\right)\left(x^2-y^2+a^2\left(3-2\sqrt{2}\right)\right)\right]\left[\left(3+2\sqrt{2}\right)\right]x^2-y^2-b^2\left(3-2\sqrt{2}\right)$$

(F.6)

G) Cross section bounded by two circles

$$\Gamma:\left(x^2+y^2-a^2\right)\left(x^2+y^2-2bx^2\right)=0 \qquad \text{(G.1)}$$

$$x^2+y^2=a^2,\ x^2+y^2=2bx,\ x^2+y^2=b^2 \qquad \text{(G.2)}$$



This as expressing and $\frac{1}{Z}$

$$\therefore \phi(Z) = bZ + \frac{a^2}{2} - \frac{a^2 b}{2}$$

$$\therefore \overline{\phi(Z)} = b\overline{Z} + \frac{a^2}{2} - \frac{a^2 b}{2} \qquad \text{(G.4)}$$

$$f(Z) = -\frac{c}{4\mu}\phi(Z) \quad \text{and} \quad \overline{f(Z)} = -\frac{c}{4\mu}\overline{\phi(Z)} \qquad \text{(G.5)}$$

The velocity is given by $w = -\frac{c}{4\mu}Z\overline{Z} + f(Z) + \overline{f(Z)}$

$\therefore$ An Expression involving Z and $\frac{1}{z}$

3. Conclusion

The steady flow of a viscous liquid through a straight tube of non-circular cross section can be solved by a complex variable method.

This method is successful whenever the boundary of the tube can be expressed as

$Z\bar{Z}$ = function of Z + its conjugate.

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References

[1] Csonka : Proceedings Of 5th Congress Of Istam (1959)
[2] Therotecal Hydro Dynamics By .L.M.Milne – Thomson, C.B.E, Macmillan Company, New York.(Chapter-Xxi, 21.44, Page-[653])
[3] Fluid Dynamics By. Frank .Chorlton, C.B.S Publishers And Distributors, Delhi,(2004) (Part-V [Chapter-14])
[4] Fluid Dynamics By .Prentice Hall, Englewood Cliffs Nj 1967)