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Hyers-Ulam Stability of Wilson's Functional Equation on Hypergroups

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Abstract: Our main goal is to study the continuous and bounded solution of the functional equations $\left\langle \delta_x * \delta_y, f \right\rangle + \left\langle \delta_x * \delta_{\check{y}}, f \right\rangle = 2f(x)g(y), \quad x, y \in X$, where $(X,*)$ is a hypergroup.

In addition, the Hyers-Ulam stability problem for this functional equation on hypergroups is considered.

Keywords: *Hypergroup, Wilson's functional equation, d'alembert's equation, Representation, Hyers-Ulam stability.*

1. INTRODUCTION

By a Wilson's functional equation on a hypergroup $(X,*)$ we will here understand the functional equation

$$\left\langle \delta_x * \delta_y, f \right\rangle + \left\langle \delta_x * \delta_{\check{y}}, f \right\rangle = 2f(x)g(y), \quad x, y \in X, \tag{1.1}$$

Where $f, g : X \to C$ are two unknown functions to be determined and $\check{y}$ is the adujugate of y in $(X,*)$.

Special cases of Wilson's functional equation on a hypergroup $(X,*)$ are d'Alembert's functional equation

$$\left\langle \delta_x * \delta_y, f \right\rangle + \left\langle \delta_x * \delta_{\check{y}}, f \right\rangle = 2f(x)f(y), \quad x, y \in X, \tag{1.2}$$

and Jensen's functional equation

$$\left\langle \delta_x * \delta_y, f \right\rangle + \left\langle \delta_x * \delta_{\check{y}}, f \right\rangle = 2f(x), \quad x, y \in X. \tag{1.3}$$

A number of results have been obtained for the equation (1.4) and the corresponding Wilson's functional equation (1.5) on groups

$$g(xy)+g(x\sigma(y))=2g(x)g(y),\ \ x,y\in G, \tag{1.4}$$

$$f(xy)+f(x\sigma(y))=2f(x)g(y),\ \ x,y\in G, \tag{1.5}$$

Where f,g are complex valued functions on a group G and $\sigma:G\to G$ be an involution of G, i.e., $\sigma(xy)=\sigma(y)\sigma(x)$ and $\sigma(\sigma(x))=x$ for all $x,y\in G$. In 2008, Th. Davison [6] proved the following result:

Let G be a topological group and $f:G\to C$ a continuous function with $f(e)=1$ satisfying $f(xy)+f(x\sigma(y))=2f(x)g(y),\ x,y\in G,$ for all x,y in G. Then there is a continuous (group) homomorphism $h:G\to SL_2(C)$ such that $f(x)=\frac{1}{2}tr(h(x)),\ x\in G.$

In [20] H. Stetkær gave solutions of (1.4) introducing the theory of representations. Precisely, he proved that the non-zero continuous solutions f of (1.4) are the functions of the form $f=\frac{1}{2}\chi_\pi$ where π ranges over the 2-dimensional continuous representations of G for which $\pi(x)\in SL_2(C)$ for all $x\in G$. For more information on the equation (1.4) and (1.5) the interested reader should refer to [2, 5, 6, 10, 16-20].

The study of functional equations on hypergroups started with some recent results. Székelyhidy [22, 23] and Orosz and Székelyhidi [11] describe moment functions, additive functions and multiplicative functions in special cases of hypergroups. In [12], sine and cosine functional equations are considered and solved on arbitrary polynomial hypergroups in a single variable and the method of solution is based on spectral synthesis. Recently the authors [14] study the abelian solutions of the d'Alembert's functional equation (1.2) and in [26], inspired by the works of Davison [6] and Stetkær [26], they proved the following structure theorem. Note that $g:X\to C$ is abelian if $\langle \delta_x*\delta_y*\delta_z,g\rangle=\langle \delta_x*\delta_z*\delta_y\rangle$ for all $x,y,z\in X$.

Theorem 1. (a) *The non-zero continuous solutions of (1.2) on a hypergroup* $(X,*)$ *are the functions of the form*

$$g(x)=\frac{1}{2}tr(\pi(x)),\ x\in X,$$

Where π *ranges over the 2-dimensional continuous representations of* G *for which* $\pi(\check{x})=adj(\pi(x))$ *for all* $x\in G$ *and*

$$adj:Mat_2(C)\to Mat_2(C)\ \ \begin{pmatrix}\alpha & \gamma\\ \delta & \beta\end{pmatrix}\to\begin{pmatrix}\beta & -\gamma\\ -\delta & \alpha\end{pmatrix}$$

b) $g=\frac{1}{2}tro\pi$ *is non-abelian if and only if* π *is irreducible*

A solution of Wilson's functional equation (1.1) is a pair $(f,g)\in C_b(X)\times C_b(X)$ satisfying (1.1). We say that the function f in the solution (f,g) is a Wilson function associated to g.

The purpose of the present paper is to study the functional equation (1.1) on hypergroups. Precisely, we determine the continuous and bounded solutions of (1.1) in the case where f is abelian function. In the case where g is non-abelian fixed solution of (1.2) we determine the space $W(g)$ of bounded and continuous function f satisfying Wilson's functional equation (1.1).

In [22] Székelyhidy, deals with the stability of exponential (i.e. multiplicative) functions on hypergroups. Precisely, he proved the following result ([22, Theorem 7.1]), which is known as a superstability:

Let X be a hypergroup and let $f, g : X \to C$ be continuous functions with the property that the function

$$y \mapsto \int_X fd(\delta_x * \delta_y) - f(x)g(y)$$

is bounded for all y in X. Then either f is bounded, or g is exponential (i.e. multiplicative function).

In last section of the present paper, we shall extend the investigation given by L. Székelyhidi ([22, Theorem 7.1]) to functional equations (1.1) and (1.2) in the case where $(X,*)$ is any topological hypergroup and f, g are continuous functions on X.

The contents of the present paper are as follows.

In the second section, we give some preliminaries on hypergroups and we prove some lemmas which will be used in the proof of our results.

In the third section, we describe the set of bounded and continuous solutions of the functional equations (1.1) on hypergroups.

In the fourth section, Hyers—Ulam stability problem for the functional equation (1.1) on hypergroups is considered. On the stability problem, the interested reader should refer to [1-3, 7, 8, 13, 15, 21, 22, 24, 25, 26].

2. PRELEMINARY RESULTS

Our notations and definitions are described in this section. We will without further mentioning keep it during the rest of the paper.

2.1 Hypergroups

We start with some notations: For a locally compact Hausdorff space X, let $M(X)$ denote the complex space of all bounded Borel measures on X, if $\mu \in M(X)$, $Supp(\mu)$ is the support of μ. The unit point mass concentrated at x is indicated by δ_x. Let $K(X)$ be the complex algebra of all continuous complex-valued functions on X with compact support and $C(X)$ (resp. $C_b(X)$ the complex algebra of all continuous (resp. continuous and bounded) complex-valued functions on X. Now, recall some basic notions and used notation from the hypergroup theory.

Definition 1. *If $M(X)$ is a Banach algebra with a associative multiplication $*$ (called a convolution), then $(X,*)$ is a hypergroup if the following axioms are satisfied*

***X**1. If μ and ν are probability measures, then so is $\mu * \nu$.*

***X**2. The mapping $(\mu,\nu) \to \mu * \nu$ is continuous from $M(X) \times M(X)$ into $M(X)$ where $M(X)$ is endowed with the weak topology with respect to $K(X)$.*

***X**3. There is an element $e \in X$ such that $\delta_e * \mu = \mu * \delta_e = \mu$ for all $\mu \in M(X)$.*

***X**4. There is a homeomorphic mapping $x \to \check{x}$ of X into itself such that $(\check{x})\check{} = x$ and $e \in Supp(\delta_x * \delta_y)$ if and only if $y = \check{x}$.*

***X**5. For all $\mu, \nu \in M(X)$, $(\mu * \nu)\check{} = \check{\nu} * \check{\mu}$ where $\check{\mu}$ is defined by*

$$\langle \breve{\mu}, f \rangle = \langle \mu, \breve{f} \rangle = \int_X f(\breve{t}) d\mu(t), \quad f \in C_b(X).$$

***X*6.** *The mapping* $(x, y) \to Supp(\delta_x * \delta_y)$ *is continuous from* $X \times X$ *into the space of compact Subsets of* X *with the topology described in [9, Sect. 2.5].*

The definitive set of axioms was given first by Jewett in his encyclopedic article [9]. A hypergroup $(X,*)$ is called commutative if its convolution is commutative.

We review some notations: Let $f \in C_b(X)$, for all $x \in X$ and $\mu \in M(X)$, we put

$$\langle \delta_x, f \rangle = f(x), \qquad \breve{f}(x) = f(\breve{x})$$

$$\langle \mu, f \rangle = \int_X \langle \delta_x, f \rangle d\mu(x),$$

If $\mu, \nu \in M(X)$ we define the convolution measure $\mu * \nu$ by

$$\langle \mu * \nu, f \rangle = \int_X \langle \delta_x * \delta_y, f \rangle d\mu(x) d\nu(y).$$

f is said to be even or invariant, (resp. odd), if $\breve{f} = f$, (resp. $\breve{f} = -f$).

Definition 2. ([4]) *Let* $(X,*)$ *be a hypergroup and* $\chi : X \to C$ *be a function, we say that*

i) χ *is a multiplicative function of* $(X,*)$ *if it has the property*

$$\langle \delta_x * \delta_y, \chi \rangle = \chi(x)\chi(y), \quad \text{for all } x, y \in X.$$

ii) χ is a hermitian function if $\chi(\breve{x}) = \overline{\chi(x)}$ for all $x \in X$.

iii) χ is a hypergroup character of $(X,*)$ if it is bounded, continuous, multiplicative and hermitian function.

Definition 3. *[9, Sect. 11.3] Let* H *be a Hilbert space,* $B(H)$ *the Banach* $*$*-algebra of all bounded linear operators on* H*, and let* I *be the identity operator.*

We say that π *is a representation of* X *on* H *if the following four conditions are satisfied:*

i) The mapping $\mu \to \pi(\mu)$ *is a* $*$*-homomorphism from* $M(X)$ *into* $B(H)$.

ii) If $\mu \in M(X)$ *then* $\|\pi(\mu)\| \le \|\mu\|$.

iii) $\pi(\delta_e) = I$.

iv) If $a, b \in H$ *then the mapping* $\mu \to \langle \pi(\mu)a, b \rangle$ *is bounded and continuous, and*

$$\langle \pi(\mu)a, b \rangle = \int_X \langle \pi(t)a, b \rangle d\mu(t), \quad \mu \in M(X).$$

Let L and R denote respectively the left and right representation of X on $C_b(X)$ i.e. $L(y)f(x) = \langle \delta_{\breve{y}} * \delta_x, f \rangle$ and $R(y)f(x) = \langle \delta_x * \delta_y, f \rangle$ for all $x, y \in X$ and $f \in C_b(X)$ Note that $L(a)$ and $R(b)$ commute for all $a, b \in X$ as is well known and also easy to check.

2.2 Auxiliary Results

Definition 4. *Let* $f : X \to C$ *be a continuous and bounded function on* X

(i) f *is said to satisfy Kannappan's type condition if*

$$\langle \mu * \nu * \varpi, f \rangle = \langle \mu * \varpi * \nu, f \rangle \text{ for all } \mu, \nu, \varpi \in M(X). \tag{2.1}$$

(ii) We say that f *is abelian if it satisfies (2.1).*

Definition 5. *Let* $(X,*)$ *be a hypergroup.*

(i) A d'Alembert function on X *is a continuous and bounded non-zero solution* $g : X \to C$ *of d'Alembert's functional equation (1.2).*

(ii) A solution of Wilson functional equation is a pair (f, g) *of functions in* $C_b(X)$ *) satisfying*

$$\langle \delta_x * \delta_y, f \rangle + \langle \delta_x * \delta_{\breve{y}}, f \rangle = 2f(x)g(y), \quad x, y \in X,$$

We say that the function f in the solution (f,g) is a Wilson function corresponding to g .

In the next lemma certain assumptions are equivalent.

Lemma 1. *Let $(X,*)$ be a hypergroup and $f,g\in C_b(X)$ then the following assertions are equivalent*

i) $\langle\delta_x*\delta_y,f\rangle+\langle\delta_x*\delta_{\check{y}},f\rangle=2f(x)g(y),$ *for all* $x,y\in X$.

ii) $\langle\mu*\nu,f\rangle+\langle\mu*\check{\nu},f\rangle=2\langle\mu,f\rangle\langle\nu,g\rangle(y),$ *for all* $\mu,\nu\in M(X)$.

Proof. i) ⇒ii) Suppose that $\langle\delta_x*\delta_y,f\rangle+\langle\delta_x*\delta_{\check{y}}\rangle=2f(x)g(y),$ for all $x,y\in X$, then

$$\begin{aligned}\langle\mu*\nu,f\rangle+\langle\mu*\check{\nu}\rangle&=\int_X\int_X\langle\delta_x*\delta_y,f\rangle d\mu(x)d\nu(y)+\int_X\int_X\langle\delta_x*\delta_{\check{y}},f\rangle d\check{\nu}(y)\\&=\int_X\int_X(\langle\delta_x*\delta_y,f\rangle+\langle\delta_x*\delta_{\check{y}}\rangle)d\mu(x)d\mu(y)\\&=2\int_X\int_X\langle\delta_x,f\rangle\langle\delta_y f\rangle d\mu(x)d\nu(y)\\&=2\langle\mu,f\rangle\langle\nu,g\rangle(y),\end{aligned}$$

The other implication is trivial.

Lemma 2. *Let $(X,*)$ be a hypergroup and $f,g\in C_b(X)$ a solution of the functional equation*

$\langle\delta_x*\delta_y,f\rangle+\langle\delta_x*\delta_{\check{y}},f\rangle=2f(x)g(y),\quad x,y\in X,$

such that $f\neq 0$ Then $g(e)=1$ and g satisfies the d'Alembert's long functional equation

$$\langle\delta_x*\delta_y,g\rangle+\langle\delta_y*\delta_x,g\rangle+\langle\delta_x*\delta_{\check{y}},g\rangle+\langle\delta_{\check{y}}*\delta_x,g\rangle=4g(x)g(y),\quad x,y\in X. \tag{2.2}$$

Proof. Choose $a\in X$ such that $f(a)\neq 0$, we get from the equation (1.1) that

$$g(x)=\frac{\langle\delta_a*\delta_x,f\rangle+\langle\delta_a*\delta_{\check{x}}\rangle}{2f(a)},$$

which implies that $g(e)=1$. Using Lemma 1 we obtain

$$\begin{aligned}&2f(a)(\langle\delta_x*\delta_y,g\rangle+\langle\delta_y*\delta_x,g\rangle+\langle\delta_x*\delta_{\check{y}},g\rangle+\langle\delta_{\check{y}}*\delta_x,g\rangle)\\&\quad=\langle\delta_a*\delta_x*\delta_y,g\rangle+\langle\delta_a*\delta_{\check{y}}*\delta_{\check{x}},g\rangle+\langle\delta_a*\delta_y*\delta_x,g\rangle+\langle\delta_a*\delta_{\check{x}}*\delta_{\check{y}},g\rangle\\&\quad+\langle\delta_a*\delta_x*\delta_{\check{y}},g\rangle+\langle\delta_a*\delta_y*\delta_{\check{x}},g\rangle+\langle\delta_a*\delta_{\check{y}}*\delta_x,g\rangle+\langle\delta_a*\delta_{\check{x}}*\delta_y,g\rangle\\&\quad=\{\langle\delta_a*\delta_x*\delta_y,g\rangle+\langle\delta_a*\delta_x*\delta_{\check{y}},g\rangle\}+\{\langle\delta_a*\delta_y*\delta_x,g\rangle+\langle\delta_a*\delta_y*\delta_{\check{x}},g\rangle\}\\&\quad+\{\langle\delta_a*\delta_{\check{x}}*\delta_y,g\rangle+\langle\delta_a*\delta_{\check{x}}*\delta_{\check{y}},g\rangle\}+\{\langle\delta_a*\delta_{\check{y}}*\delta_x,g\rangle+\langle\delta_a*\delta_{\check{y}}*\delta_{\check{x}},g\rangle\}\\&\quad=2\{\langle\delta_a*\delta_x,f\rangle+\langle\delta_a*\delta_{\check{x}},f\rangle\}g(y)+2\{\langle\delta_a*\delta_y,f\rangle+\langle\delta_a*\delta_{\check{y}},f\rangle\}g(x)\\&\quad=4f(a)g(x)g(y)+4f(a)g(y)g(x)=8f(a)g(x)g(y).\end{aligned}$$

As $f(a)\neq 0$ we get

$\langle\delta_x*\delta_y,g\rangle+\langle\delta_y*\delta_x,g\rangle+\langle\delta_x*\delta_{\check{y}},g\rangle+\langle\delta_{\check{y}}*\delta_x,g\rangle=4g(x)g(y)$ *for all* $x,y\in X$.

Hence g is a solution of the equation (2.2).

Note that if (f,g) is a solution of (1.1) such that f is abelian then g is an abelian d'Alembert function.

Remark 1. Let the pair (f,g) be a solution of the functional equation (1.1) in $(X,*)$ such that $f \neq 0$. If f is abelian then so is g.

Let f be a complex valued function on X. We denote by f_e and f_o respectively the even and the odd part of f i. e. $f_e = \frac{f+\breve{f}}{2}$ and $f_o = \frac{f-\breve{f}}{2}$ in the following lemma we give some properties of solutions of the functional equation (1.1).

Lemma 3. *Let $(f,g) \in C_b(X) \times C_b(X)$ be a solution of the equation (1.1) such that $f \neq 0$. Then*

i) $f_e = f(e)g$. In particular, we conclude that f is odd if and only if $f(e) = 0$.

ii) If f is odd then

$\langle \mu * \nu, f \rangle + \langle \nu * \mu, f \rangle = 2\langle \mu, f \rangle \langle \nu, g \rangle + 2\langle \nu, f \rangle \langle \mu, g \rangle$ *for all* $\mu, \nu \in M(X)$.

iii) If g is a d'Alembert function then f_e and f_o are Wilson functions associated to g.

Proof. By posing $\mu = \delta_e$ in (1.1) we find that $f_e = f(e)g$. This is the case i). For the case ii) exchanging μ and ν in (1.1) and adding the result to (1.1). The result, thus, comes from the fact that f is odd. Finally, if we assume that g is a d'Alembert function, by using the expression $f_e = \frac{f+\breve{f}}{2} = f(e)g$ we have f_e is a Wilson function. Similarly, by the equality

$f_o = \frac{f-\breve{f}}{2} = f - f(e)g$,

we find immediately that f_o is also a Wilson function associated to g.

Now, we study the bounded and continuous solutions of (1.1) by examining separately the abelian case and the non-abelian case.

3. Wilson's Functional Equation (1.1)

3.1 The abelian case for the equation (1.1).

In this section, we determine all solutions $(f,g) \in C_b(X) \times C_b(X)$ of the equation (1.1) such that f is abelian. These solutions will be expressed in terms of multiplicative functions χ on X and solutions of the functional equation

$$\langle \delta_x * \delta_y, f \rangle = f(x)\chi(y) + \chi(x)f(y), \quad x, y \in X.$$

Proposition 1. *Let $(f,g) \in C_b(X) \times C_b(X)$ such that g is abelian and f is central, i.e. $\langle \delta_x * \delta_y, f \rangle = \langle \delta_y * \delta_x, f \rangle$ for all $x, y \in X$. If (f,g) is a solution of the equation (1.1) then:*

i) There is a multiplicative function χ on X such that $g = \frac{\chi + \breve{\chi}}{2}$.

ii) If $\chi \neq \breve{\chi}$ then there exist two complex constants α and β such that

$$f = \alpha \frac{\chi - \breve{\chi}}{2} + \beta \frac{\chi + \breve{\chi}}{2}.$$

iii) If $\chi = \breve{\chi}$ then there exists a complex constant γ such that $f = \gamma\chi + l$ where l is a solution of the functional equation

$$\langle \delta_x * \delta_y, l \rangle = l(x)\chi(y) + \chi(x)l(y), \quad x, y \in X,$$

satisfying $\breve{l} = -l$.

Proof. i) By Lemma 2, g is a solution of the d'Alembert's long functional equation (2.2). Since g is abelian, then it is central hence it is a solution of equation (1.2). Using [14, Corollary 1] we obtain i).

ii) We decompose f into its even and odd parts $f = f_e + f_o$. The functions f_e and f_o are Wilson's functions (Lemma 3). Since f is central then so are f_e and f_o. So it suffices to prove the proposition for f_e and f_o separately.

Lemma 3 tells us that the even part f_e is proportional to $g = \frac{\chi + \breve{\chi}}{2}$ Which proves the proposition for f_e On the other hand f_o is a central Wilson function associated to g. By Lemma 3 we have

$$\langle \delta_x * \delta_y, f_o \rangle = f_o(x)g(y) + g(x)f_o(y), \quad x, y \in X.$$

i.e. the pair (f_o, g) is a solution of the sine addition formula [25, Corollary 26.1]. The case $f_o = 0$ is trivial. If $f_o \neq 0$, by examining the different possibilities (i)-(iv) listed in [25, Corollary 26.1] we get.

If $\chi \neq \breve{\chi}$,then there exist complex constants α and β such that

$$f = \alpha \frac{\chi - \breve{\chi}}{2} + \beta \frac{\chi + \breve{\chi}}{2},$$

and if $\chi = \breve{\chi}$, then there exists a complex constant γ such that $f = \gamma\chi + l$ where l is a solution of the functional equation

$$\langle \delta_x * \delta_y, l \rangle = l(x)\chi(y) + \chi(x)l(y), \quad x, y \in X.$$

Satisfying $\breve{l} = -l$.

Conversely, by easy calculations, we can verify that the functions presented in Proposition 1 are solutions of the equation (1.1).

Theorem 2. *Let* $f, g \in C_b(X) - \{0\}$ *such that* f *is abelian. If* (f, g) *is a solution of the equation (1.1) then:*

i) There is a multiplicative function χ *on* X *such that* $g = \frac{\chi + \breve{\chi}}{2}$.

ii) If $\chi \neq \breve{\chi}$ *then there exist two complex constants* α *and* β *such that*

$$f = \alpha \frac{\chi - \breve{\chi}}{2} + \beta \frac{\chi + \breve{\chi}}{2}.$$

iii) If $\chi = \breve{\chi}$ *then there exists a complex constant* γ *such that* $f = \gamma\chi + l$ *where* l *is a solution of the functional equation*

$$\langle \delta_x * \delta_y, l \rangle = l(x)\chi(y) + \chi(x)l(y), \quad x, y \in X,$$

satisfying $\breve{l} = -l$.

Proof. Let (f, g) be a solution of the equation (1.1). Since f is abelian then it is a central function. Let $a \in X$ such that $f(a) \neq 0$. By using the equation (1.1) we obtain the equality

$$g(x) = \frac{\langle \delta_a * \delta_x, f \rangle + \langle \delta_a * \delta_{\breve{x}} \rangle}{2f(a)}, \tag{3.1}$$

from which we find that g is abelian. Using Proposition 1 we get i) ,ii) and iii).

As consequence of Theorem 2 we have the following result on the Wilson Pexider's functional equation.

Corollary 1. *Let* $(X,*)$ *be a hypergroup and* $f, g, h \in C_b(X) - \{0\}$ *such that* f *is abelian. If* (f, g, h) *is a solution of the equation*

$$\langle \delta_x * \delta_y, f \rangle + \langle \delta_x * \delta_{\breve{y}}, f \rangle = 2g(x)h(y), \quad x, y \in X, \tag{3.2}$$

then

i) There exist $\alpha \in C - \{0\}$, $\beta, \gamma \in C$ *and multiplicative function* $\chi \neq \breve{\chi}$ *on* $(X,*)$ *such that*

$$h = \alpha\frac{\chi + \breve{\chi}}{2}, \quad f = \beta\frac{\chi + \breve{\chi}}{2} + \gamma\frac{\chi - \breve{\chi}}{2} \text{ and } g = \frac{\beta}{\alpha}\frac{\chi + \breve{\chi}}{2} + \frac{\gamma}{\alpha}\frac{\chi - \breve{\chi}}{2}.$$

ii) There exist a multiplicative function $\chi = \breve{\chi}$ *and* $(\alpha, \gamma) \in C - \{0\} \times C$ *such that*

$$h = \alpha\chi, \quad f = \gamma\chi + l \text{ and } g = \frac{f}{\alpha},$$

where l *is a solution of the functional equation*

$$\langle \delta_x * \delta_y, l \rangle = l(x)\chi(y) + \chi(x)l(y), \quad x, y \in X,$$

satisfying $\breve{l} = -l$.

Proof. If we put $y = e$ in (3.2) we find that

$$f(x) = g(x)h(e).$$

The inequality $f \neq 0$ implies $h(e) \neq 0$. Using the equation (3.2) we get that

$$\langle \delta_x * \delta_y, f \rangle + \langle \delta_x * \delta_{\breve{y}}, f \rangle = 2f(x)\frac{h(x)}{h(e)}, \quad x, y \in X,$$

So, the pair $(f, \frac{h}{h(e)})$ is a solution of the equation (1.1). Then the rest of the proof follows immediately from Theorem 2.

In the following corollary we solve the Jensen's functional equation in a hypergroup in the case where f is central.

Corollary 2. *Let* $f \in C_b(X) - \{0\}$ *such that* f *is central. If* f *is a solution of the Jensen's functional equation*

$$\langle \delta_x * \delta_y, f \rangle + \langle \delta_x * \delta_{\breve{y}}, f \rangle = 2f(x), \quad x, y \in X, \tag{3.3}$$

Then, there exists a complex constant γ such that $f = \gamma + l$ *where* l *is an odd and additive function on* $(X,*)$ *i. e.* l *satisfies the equalities:* $\langle \delta_x * \delta_y, l \rangle = l(x) + l(y), \; x, y \in X$ *and* $\breve{l} = -l$.

Proof. Let $f \in C_b(X) - \{0\}$ be a solution of the equation (3.3) then $(1, f)$ is a solution of the equation (1.1). We note that $g \equiv 1$ is an abelian d'Alembert function and if $g = 1 = \frac{\chi + \breve{\chi}}{2}$ for a multiplicative function χ on X then, since the set of multiplicative functions on X is linearly independent [14, Proposition 3] then $\chi = \breve{\chi} = 1$. By examining the assertions (i) - (iii) of Theorem 2 we deduce that the solutions of (3.3) are exactly the functions f of the form: $f = \gamma + l$ where γ is a complex constant and l is an odd additive function on X .

3.2 **The non abelian case for the equation (1.1)**

Let g be a non-abelian fixed solution of (1.2). In the following theorem, we study the space $W(g)$ of Wilson functions corresponding to g . The result is closely related to and inspired by the works by Stetkær [20].

Theorem 3. *Let* $g(x) = \frac{1}{2} tr(\pi(x)), \; x \in X$ *, be a non-abelian d'Alembert function on* X *as described in Theorem 1. We may assume that* π *is a representation of* X *on* C^2*.The corresponding space* $W(g)$ *of Wilson's functions consists of the functions of the form*

$$f(x) = \frac{1}{2} tr(A\pi(x)) \quad \textit{for all } x \in X, \tag{3.4}$$

where A *ranges over the complex 2×2 matrices.*

Proof. According to [26, Theorem 5(c)], $W(g) = span\{L(\mu)g, \mu \in M(X)\}$ where L is the left regular representation of X on $W(g)$. So any Wilson function $f \in W(g)$ has the form

$$f = \sum_{i=1}^{n} a_i L(\breve{\mu}_i) g,$$

where $a_1, a_2, ..., a_n \in C$ and $\mu_1, \mu_2, ..., \mu_n \in M(X)$. Now, for all $x \in X$ we have

$$f(x)=\sum_{i=1}^{n}a_i(L(\breve{\mu}_i)g)(x)$$

$$=\sum_{i=1}^{n}a_i\langle\mu_i*\delta_x,g\rangle=\frac{1}{2}\sum_{i=1}^{n}a_i tr\pi(\mu_i*\delta_x)$$

$$=\frac{1}{2}\sum_{i=1}^{n}a_i tr\pi(\mu_i)tr(x)$$

$$=\frac{1}{2}tr(\sum_{i=1}^{n}a_i\pi(\mu_i)tr(x),$$

so as A we may take $A=\sum_{i=1}^{n}a_i\pi(\mu_i)$ we have thus proved that W(g) is contained in the space of functions of the form (3.4). Now $W(g)$ has dimension 4 by [26, Theorem 5(a)]. But so has the space of functions of the form (3.4), because the space of complex 2×2-matrices has dimension 4. Then we have the desired equality.

By using Theorem 3 we obtain the following result

Corollary 3. *Let $(X,*)$ be a hypergroup and $g:X\to C$ be a non-Abelian d'Alembert function on X. If $f:X\to C$ is a Wilson function corresponding to g (i.e. $f\in W(g)$), then there exists a continuous multiplicative function $\phi:(X,*)\to Mat_2(C)$ and $A\in Mat_2(C)$ satisfying $\varphi(\breve{x})=adj(\varphi(x)),\ x\in X$, such that*

$$g(x)=\frac{1}{2}tr\varphi(x)\quad and\quad f=\frac{1}{2}tr(A\varphi(x)),\ x\in X,$$

where $Mat_2(C)$ is the space of complex matrices of order 2 and

$$adj:Mat_2(C)\to Mat_2(C)\ \begin{pmatrix}\alpha & \gamma\\ \delta & \beta\end{pmatrix}\to\begin{pmatrix}\beta & -\gamma\\ -\delta & \alpha\end{pmatrix}.$$

4. STABILITY OF THE EQUATION (1.1)

There is a strong stability phenomenon which is known as a super stability. An equation of homomorphism is called superstable if each approximate homomorphism is actually a true homomorphism. This property was first observed by J. Baker, J. Lawrence, and F. Zorzitto [3] in the following Theorem: Let V be a vector space. If a function $f:V\to IR$ satisfies the inequality $|f(x+y)-f(x)f(y)|\le\varepsilon$ for some $\varepsilon>0$ and for all $x,y\in V$. Then either f is a bounded function or $f(x+y)=f(x)+f(y),\ x,y\in V$.

Later this result was generalized by J. Baker [2] and L. Székelyhidi [21, 22].

In present section we shall extend the investigation given by J. Baker [2], L. Székelyhidi [21, 22], R. Badora [1] and E. Elqorachi and M. Akkouchi [7] to equations (1.1) and (1.2).

Lemma 4. *Let $\delta>0$ be given. Assume that continuous functions $f,g:X\to C$ satisfy the inequality*

$$\left|\left\langle \delta_x * \delta_y, f\right\rangle + \left\langle \delta_x * \delta_{\bar{y}}, f\right\rangle - 2f(x)g(y)\right| \le \delta, \quad x, y \in X, \tag{4.1}$$

such that $f \neq 0$. *If* g *is unbounded then so is* f.

Proof. Assume that g is unbounded function satisfying the inequality (4.1). If $f \neq 0$ is bounded, let $M = \sup f$ and choose $a \in X$ such that $f(a) \neq 0$ then we get from the inequality (4.1) that

$$\left|\left\langle \delta_a * \delta_y, f\right\rangle + \left\langle \delta_a * \delta_{\bar{y}}, f\right\rangle - 2f(a)g(y)\right| \le \delta, \quad y \in X,$$

from which we obtain that

$$\left|2f(a)g(y)\right| - \left|\left\langle \delta_a * \delta_y, f\right\rangle + \left\langle \delta_a * \delta_{\bar{y}}, f\right\rangle\right| \le \delta, \quad y \in X,$$

so we conclude that

$$\left|g(y)\right| \le \frac{1}{2f(a)}(\delta + 2M) \quad \textit{for all } y \in X,$$

then g is bounded which contradicts our assumption.

In theorem 4 below, the Hyers--Ulam stability of equation (1.1) will be investigated without the additional condition that f satisfies the Kannappan's type condition:

$$\left\langle \delta_x * \delta_y * \delta_z, f\right\rangle = \left\langle \delta_x * \delta_z * \delta_y, f\right\rangle \quad \textit{for all} \quad x, y, z \in X.$$

Theorem 4. *Let* $\delta > 0$ *be given. Assume that continuous functions* $f, g : X \to C$ *satisfy the inequality*

$$\left|\left\langle \delta_x * \delta_y, f\right\rangle + \left\langle \delta_x * \delta_{\bar{y}}, f\right\rangle - 2f(x)g(y)\right| \le \delta, \quad x, y \in X,$$

then:

- *i)* f, g *are bounded or*
- *ii)* f *is unbounded and* g *satisfies the d'Alembert's long equation (2.3) or*
- *iii)* g *is unbounded and* f *satisfies the equation (1.1) (if* $f \neq 0$ *then* g *satisfies the d'Alembert's long equation (2.2).*

Proof. Assume that f, g satisfy inequality (4.1). ii) First we consider the case of f unbounded. For all $x, y, z \in X$ we have

$$\left|2f(z)\right|\left|\left\langle \delta_x * \delta_y, g\right\rangle + \left\langle \delta_y * \delta_x, g\right\rangle + \left\langle \delta_x * \delta_{\bar{y}}, g\right\rangle + \left\langle \delta_{\bar{y}} * \delta_x, g\right\rangle - 4g(x)g(y)\right|$$

$$= \left|2f(z)\left\{\left\langle \delta_x * \delta_y, g\right\rangle + \left\langle \delta_y * \delta_x, g\right\rangle + \left\langle \delta_x * \delta_{\bar{y}}, g\right\rangle + \left\langle \delta_{\bar{y}} * \delta_x, g\right\rangle\right\} - 8f(z)g(x)g(y)\right|$$

$$= \left|\int_X 2f(z)g(t)d(\delta_x * \delta_y)(t) + \int_X 2f(z)g(t)d(\delta_y * \delta_x)(t) + \int_X 2f(z)g(t)d(\delta_x * \delta_{\bar{y}})(t)\right|$$

$$+\left|\int_X 2f(z)g(t)d(\delta_{\check{y}}*\delta_x)(t)-8f(z)g(x)g(y)\right|$$

$$\leq\left|\int_X(\langle\delta_z*\delta_t,f\rangle+\langle\delta_z*\delta_{\check{t}},f\rangle-2f(z)g(t))d(\delta_x*\delta_y)(t)\right|$$

$$+\left|\int_X(\langle\delta_z*\delta_t,f\rangle+\langle\delta_z*\delta_{\check{t}},f\rangle-2f(z)g(t))d(\delta_y*\delta_x)(t)\right|$$

$$+\left|\int_X(\langle\delta_z*\delta_t,f\rangle+\langle\delta_z*\delta_{\check{t}},f\rangle-2f(z)g(t))d(\delta_x*\delta_{\check{y}})(t)\right|$$

$$+\left|\int_X(\langle\delta_z*\delta_t,f\rangle+\langle\delta_z*\delta_{\check{t}},f\rangle-2f(z)g(t))d(\delta_{\check{y}}*\delta_x)(t)\right|$$

$$+\left|\int_X(\langle\delta_t*\delta_y,f\rangle+\langle\delta_t*\delta_{\check{y}},f\rangle-2f(t)g(y))d(\delta_z*\delta_x)(t)\right|$$

$$+\left|\int_X(\langle\delta_t*\delta_x,f\rangle+\langle\delta_t*\delta_{\check{x}},f\rangle-2f(t)g(x))d(\delta_z*\delta_y)(t)\right|$$

$$+\left|\int_X(\langle\delta_t*\delta_x,f\rangle+\langle\delta_t*\delta_{\check{x}},f\rangle-2f(t)g(x))d(\delta_z*\delta_{\check{y}})(t)\right|$$

$$+\left|\int_X(\langle\delta_t*\delta_y,f\rangle+\langle\delta_t*\delta_{\check{y}},f\rangle-2f(t)g(y))d(\delta_z*\delta_{\check{x}})(t)\right|$$

$$+2|g(y)||\langle\delta_z*\delta_x,g\rangle+\langle\delta_z*\delta_{\check{x}},g\rangle-2f(x)g(x)|$$

$$+2|g(x)||\langle\delta_z*\delta_y,g\rangle+\langle\delta_z*\delta_{\check{y}},g\rangle-2f(z)g(y)|.$$

By virtue of inequality (4.1), we obtain

$$|2f(z)||\langle\delta_x*\delta_y,g\rangle+\langle\delta_y*\delta_x,g\rangle+\langle\delta_x*\delta_{\check{y}},g\rangle+\langle\delta_{\check{y}}*\delta_x,g\rangle-4g(x)g(y)|$$

$$\leq 8\delta+(2|g(y)|+2|g(x)|)\delta. \tag{4.2}$$

Since f is unbounded, from the preceding (4.2), we conclude that g is a solution of the d'Alembert's long equation (2.2), which ends the proof in this case.

iii)If g is unbounded, then $f=0$ is a trivial solution of the Wilson equation (1.1). Now assume that $f\neq 0$.

$$|2g(z)||\langle\delta_x*\delta_y,f\rangle+\langle\delta_x*\delta_{\check{y}},f\rangle-2f(x)g(y)|$$

$$=\left|\int_X 2g(z)f(t)d(\delta_x*\delta_y)(t)+\int_X 2g(z)f(t)d(\delta_x*\delta_{\bar{y}})(t)-4f(x)g(y)g(z)\right|$$

$$\leq\left|\int_X (\langle\delta_t*\delta_z,f\rangle+\langle\delta_t*\delta_{\bar{z}},f\rangle-2f(t)g(z))d(\delta_x*\delta_y)(t)\right|$$

$$+\left|\int_X (\langle\delta_t*\delta_z,f\rangle+\langle\delta_t*\delta_{\bar{z}},f\rangle-2f(t)g(z))d(\delta_x*\delta_{\bar{y}})(t)\right|$$

$$+\left|\int_X (\langle\delta_x*\delta_t,f\rangle+\langle\delta_x*\delta_{\bar{t}},f\rangle-2f(x)g(t))d(\delta_y*\delta_x)(t)\right|$$

$$+\left|\int_X (\langle\delta_x*\delta_t,f\rangle+\langle\delta_x*\delta_{\bar{t}},f\rangle-2f(x)g(t))d(\delta_y*\delta_{\bar{z}})(t)\right|$$

$$+\left|\int_X (\langle\delta_x*\delta_t,f\rangle+\langle\delta_x*\delta_{\bar{t}},f\rangle-2f(x)g(t))d(\delta_{\bar{z}}*\delta_y)(t)\right|$$

$$+\left|\int_X (\langle\delta_x*\delta_t,f\rangle+\langle\delta_x*\delta_{\bar{t}},f\rangle-2f(x)g(t))d(\delta_z*\delta_y)(t)\right|$$

$$+\left|\int_X (\langle\delta_t*\delta_y,f\rangle+\langle\delta_t*\delta_{\bar{y}},f\rangle-2f(t)g(y))d(\delta_x*\delta_{\bar{z}})(t)\right|$$

$$+\left|\int_X (\langle\delta_t*\delta_y,f\rangle+\langle\delta_t*\delta_{\bar{y}},f\rangle-2f(t)g(y))d(\delta_x*\delta_z)(t)\right|$$

$$+|2f(x)|\left|\langle\delta_y*\delta_z,g\rangle+\langle\delta_y*\delta_{\bar{z}},g\rangle+\langle\delta_z*\delta_y,g\rangle+\langle\delta_{\bar{z}}*\delta_y,g\rangle-4g(y)g(z)\right|$$

$$+\left|2\langle\delta_x*\delta_z,f\rangle g(y)+2\langle\delta_x*\delta_{\bar{z}},f\rangle g(y)-4f(x)g(z)g(y)\right|$$

In virtue of inequality (4.1), we obtain

$$|2g(z)|\left|\langle\delta_x*\delta_y,f\rangle+\langle\delta_x*\delta_{\bar{y}},f\rangle-2f(x)g(y)\right|\leq 8\delta+2|g(y)|\delta$$

$$+|2f(x)|\left|\langle\delta_y*\delta_z,g\rangle+\langle\delta_y*\delta_{\bar{z}},g\rangle+\langle\delta_z*\delta_y,g\rangle+\langle\delta_{\bar{z}}*\delta_y,g\rangle-4g(y)g(z)\right|.$$

By using Lemma 4 we see that g is unbounded implies necessarily that so is f hence according to theorem 4 ii) g is a solution of the equation (2.2). We conclude that

$$|2g(z)|\left|\langle\delta_x*\delta_y,f\rangle+\langle\delta_x*\delta_{\bar{y}},f\rangle-2f(x)g(y)\right|\leq 8\delta+2|g(y)|\delta. \tag{4.3}$$

Since g is unbounded, from the preceding (4.3), we conclude that f satisfies the equation (1.1). If $f\neq 0$ then, by Lemma 2, g satisfies the equation (2.2) and the proof of the theorem 4 is finished.

As a consequence of Theorem 4, we have the following result on the superstability of the d'Alembert equation (1.2) which generalizes the Baker's result on the classical d'Alembert functional equation on an abelian group [2] (Theorem 5).

Corollary 4. Let $\delta > 0$ be given. Assume that continuous functions $f : X \to C$ satisfies the inequality

$$\left| \left\langle \delta_x * \delta_y, f \right\rangle + \left\langle \delta_x * \delta_{\check{y}}, f \right\rangle - 2 f(x) f(y) \right| \le \delta, \quad x, y \in X,$$

then either: $$|f(x)| \le \frac{1+\sqrt{1+2\delta}}{2}, \quad x \in X,$$

or $$\left\langle \delta_x * \delta_y, f \right\rangle + \left\langle \delta_x * \delta_{\check{y}}, f \right\rangle = 2 f(x) f(y), \quad x, y \in X.$$

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