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On Differential Inequalities for Discontinuous Nonlinear Integro-Differential Equations

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Abstract: *In this paper existence of extremal solutions for nonlinear second order integro-differential equations with discontinuous right hand side is obtained under certain monotonicity conditions and without assuming the existence of upper and lower solutions. Two basic differential inequalities corresponding to these integro-differential equations are obtained in the form of extremal solutions. And also we prove uniqueness of solutions of given integro-differential equations.*

Keywords: *Complete lattice, Tarski fixed point theorem, isotone increasing, minimal and maximal solutions.*

1. INTRODUCTION

In [1], B C Dhage ,G P Patil established the existence of extremal solutions of the nonlinear two point boundary value problems(BVPs for short) and in [2] , B C Dhage established the existence of weak maximal and minimal solutions of the nonlinear two point boundary value problems with discontinuous functions on the right hand side. We have develop the results of [1]and [2] , to nonlinear systems of equations [3].We now extend the results of [1],[2]and [3]to second order nonlinear integro-differential equations.

2. INTEGRO -DIFFERENTIAL EQUATIONS

Let R denote the real line and R^+, the set of all non negative real numbers. Suppose J= [a, b] is a closed and bounded interval in R. In this paper we shall establish the existence of maximal and minimal solutions for the nonlinear second order integro-differential equations of the form

$$-x^{11} = f(t, x, x^1, T^* x) \quad a.a.\ t \in J, \tag{2.1}$$

$$x(a)=0=x^1(b) \tag{2.2}$$

where $T^* x(t) = \int_a^b G(t,s)x(s)ds$ for $t \in J$, $x^1 = \dfrac{dx}{dt},\ x^{11} = \dfrac{d^2x}{dt^2}$

and $f : J \times R \times R \times R \to R$ is a function , $G(t,s)$ is Green’s function of homogeneous boundary value problem

$$-x^{11} = 0 \quad , \quad x(a)=0=x^1(b).$$

It is known (see [4, p.191] and [5, p.25]) that Green’s function is continuous and nonnegative.

By a solution x of the BVP (2.1)–(2.2), we mean a function $x : J \to R$ whose first derivative exists and is absolutely continuous on J, satisfying(2.1)–(2.2).

It follows from the theory of Green's function and superposition principle that the BVP(2.1)–(2.2) is equivalent to the following integral equation

$$x(t)=\int_a^b G(t,s)\,f(s,x(s),x^1(s),T^*x(s))\,ds\,,\ t\in J\,. \tag{2.3}$$

Then $x^1(t)=\int_t^b f(s,x(s),x^1(s),T^*x(s))\,ds$, $t\in J$.

To prove the main existence result we need the following preliminaries.

Let C(J,R) denote the space of continuous real valued functions on J , M(J,R) the space of all measurable real valued functions on J and B(J,R), the space of all bounded real valued functions on J . By BM(J,R),we mean the space of bounded and measurable real valued functions on J, where J is given interval . We define an order relation $\leq$ in BM(J, R) by $x,y\in$ BM(J,R), then $x\leq y$ if and only if $x(t)\leq y(t)$ and $x^1(t)\leq y^1(t)$ for all $t\in J$. A set S in BM(J,R) is a complete lattice w.r.t $\leq$ if supremum and infimum of every sub set of S exists in S.

Definition 2.1 A mapping T: BM(J,R) $\rightarrow$ BM(J,R) is said to be isotone increasing *if* $x,y\in$ BM (J,R) with $x\leq y$ implies T$x\leq$ Ty.

The following fixed point theorem due to Tarski [6] will be used in proving the existence of extremal solutions for nonlinear second order integro-differential equations.

Definition 2.2 A solution $x_M(t)$ of boundary value problem (2.1)-(2.2) is said to be maximal if u is any other solution of (2.1)-(2.2) , then $u(t)\leq x_M(t)$ for all $t\in J$. A minimal solution x_m of boundary value problem (2.1)-(2.2) may be defined in a similar manner.

Theorem 2.3 Let E be a nonempty set and let T_1: E $\rightarrow$ E be a mapping such that

(i) (E, $\leq$) is a complete lattice

(ii) T_1 is isotone increasing and

(iii) F= {u $\in$ E / T_1u=u}.

Then F is non empty *and* $(F,\leq)$ is a complete lattice.

For proving the main existence result, define a norm on BM (J,R) by

$$\|x\|_1=\frac{8}{3(b-a)^2}\sup_{t\in J}|x(t)|+\frac{1}{3(b-a)}\sup_{t\in J}|x^1(t)|+\frac{16}{3(b-a)^4}\sup_{t\in J}|T^*x(t)|$$

for $x\in BM(J,R)$.

Clearly BM(J,R) is a Banach space with the above norm .We shall now prove the existence of maximal and minimal solutions of (2.1)–(2.2). For this we need the following assumptions:

(g_1) f is bounded on J x R x R x R by η , η >0 .

(g_2) f $(t,\varphi(t),\varphi^1(t),T^*\varphi(t))$ is a Lebesgue measurable function for all Lebesgue measurable functions φ , φ^1, $T^*\varphi$ on J , and

(g_3) $f(t,x,x^1,T^*x)$ is nondecreasing in x, x^1 and T^*x in R for *a.a.* $t\in J$.

Theorem 2.4 Assume that the hypotheses (g_1)- (g_3) hold . Then the BVP (2.1)-(2.2) has maximal and minimal solutions on J.

Proof. Define a subset U of the Banach space BM (J, R) by

U= $\left\{ x \in BM(J,R) : \|x\|_1 \le \eta \right\}$ (2.4)

Clearly U is a closed, convex and bounded subset of the Banach space BM(J,R) and we see that (U , $\le$) is a complete lattice. Now define an operator $Q : \text{U} \rightarrow$ BM(J,R) by

$$Qx(t) = \int_a^b G(t,s)\ f(s, x(s), x^1(s), T^* x(s))\, ds \ ,\ t \in J \tag{2.5}$$

And

$$Q\,T^* x(t) = \int_a^b G(t,s) x(s) ds\,.$$

Then

$$Qx^1(t) = \int_t^b f(s, x(s), x^1(s), T^* x(s))\, ds \ ,\ t \in J\,.$$

Obviously the functions (Qx), (Qx^1) and (Q(T$^*x$)) are continuous on J and hence Lebesgue measurable on J. We shall now show that Q maps U into itself .Let $x \in$ U be an arbitrary point.

Then

$$\left|Q\ x(t)\right| \le \int_a^b G(t,s) \left| f(s, x(s), x^1(s), T^* x(s)) \right| ds \qquad \le \eta\ \frac{(b-a)^2}{8}\ .$$

And

$$\left|Q\ x^1(t)\right| \le \int_t^b \left| f(s, x(s), x^1(s), T^* x(s)) \right| ds \qquad \le \eta\ (b-a)\,.$$

Also

$$\left|Q\ \left(T^* x(t)\right)\right| \le \int_a^b G(t,s) \left| x(s) \right| ds \ . \tag{2.6}$$

We have

$$\left|x(s)\right| \le \int_a^b (s-a) \left| f(s, x(s), x^1(s), T^* x(s)) \right| ds\,.$$

This implies that

$$\left|x(s)\right| \le \eta\ \frac{(b-a)^2}{2}\,.$$

Then by (2.6), we have

$$\left|Q\,T^* x(t)\right| \le \int_a^b G(t,s) \eta\ \frac{(b-a)^2}{2} ds \qquad \le \eta\ \frac{(b-a)^2}{2}\ \frac{(b-a)^2}{8}\ .$$

Therefore

$$\|Q\,x\|_1 = \frac{8}{3(b-a)^2}\sup_{t\in J}|Q\,x(t)| + \frac{1}{3(b-a)}\sup_{t\in J}|Q\,x^1(t)| + \frac{16}{3(b-a)^4}\sup_{t\in J}|Q\,T^*x(t)|$$

$$\le \frac{8}{3(b-a)^2}\,\eta\,\frac{(b-a)^2}{8} + \frac{1}{3(b-a)}\,\eta\,(b-a) + \frac{16}{3(b-a)^4}\eta\,\frac{(b-a)^2}{2}\frac{(b-a)^2}{8}$$

$$\le \eta\ .$$

This shows that Q maps U into itself. Again let x, y be such that $x \le y$, then by (g_3), we have

$$Q\,x(t) = \int_a^b G(t,s)\,f(s,x(s),x^1(s),T^*x(s))ds$$

$$\le \int_a^b G(t,s)\,f(s,y(s),y^1(s),T^*y(s))ds = Qy(t) \text{ for } t\in J\ .$$

And

$$Q\,x^1(t) = \int_t^b f(s,x(s),x^1(s),T^*x(s))ds$$

$$\le \int_t^b f(s,y(s),y^1(s),T^*y(s))ds = Q\,y^1(t)\ ,t\in J.$$

Also

$$QT^*x(t) = \int_a^b G(t,s)x(s)ds \le \int_a^b G(t,s)y(s)ds = QT^*y(t) \quad \text{for } t\in J.$$

This shows that Q is isotone increasing on U. Now by using Theorem 2.3, we conclude that, the boundary value problem (2.1)-(2.2) has maximal and minimal solutions in U.

Finally in view of the definition of the operator Q, it follows that these extremal solutions are in C(J,R).

This completes the proof.

Now we shall show that the maximal and minimal solutions of the BVP (2.1)-(2.2) serve as the bounds for the solutions of the differential inequalities relative to the BVP (2.1)-(2.2).

Theorem 2.5 Assume that $(g_1)-(g_3)$ hold. Further if there is a function $u \in U$, where U is defined as in (2.4), such that

$$-u^{11} \le f(t, u(t),u^1(t),T^*u(t))\ a.a.\ t\in J \tag{2.7}$$

$$u(a) = 0 = u^1(b) \tag{2.8}$$

Then there exists a maximal solution x_M of the BVP (2.1)-(2.2) such that

$$u(t) \le x_M(t)\ \ \forall t\in J\ .$$

Proof. Let p=sup U. Clearly the element p exists, since U is a complete lattice. Consider the lattice interval [u,p] in U, where u is a solution of (2.7)- (2.8) .We notice that [u,p] is a complete lattice .Define an operator $Q:[u,p]\to U$ as given in (2.5) .

It can be shown as in the proof of Theorem 2.4 that Q is isotone increasing on [u,p] . We now show that Q maps [u,p] into itself . For it suffices to show that if $x\in U$ with $u \le x$ implies $u \le Qx$ .Now from the inequality (2.7)- (2.8) , it follows that

$$u^1(t) \le \int_t^b f(s,u(s),u^1(s),T^*u(s))ds$$

$$\le \int_t^b f(s,x(s),x^1(s),T^*x(s))ds \;=\; (Qx(t))^1 \quad \text{for } t \in J\,.$$

And

$$u(t) \le \int_a^b G(t,s) f(s,u(s),u^1(s),T^*u(s))ds$$

$$\le \int_a^b G(t,s) f(s,x(s),x^1(s),T^*x(s))ds \;=\; Qx(t) \quad \text{for all } t \in J\,.$$

Also

$$T^*u\,(t) = \int_a^b G(t,s)u(s)ds$$

$$\le \int_a^b G(t,s)x(s)ds = QT^*x(t) \quad \text{for } \; t \in J\,.$$

This shows that Q maps [u,p] into itself. As an application of Theorem 2.3, it follows that there is a maximal solution x_M of the integral equation (2.3) and consequently of the BVP (2.1)-(2.2) in [u, p].

Hence $u(t) \le x_M(t)$ for $t \in J$.

This completes the proof.

Theorem 2.6 Suppose that $(g_1)-(g_3)$ hold. Further if there is a function $v \in \mathrm{U}$, where U is as given in (2.4), such that

$-v^{11} \ge f(t,v,v^1,T^*v)$ *a .a .* $t \in J$,

$v(a)=0=v^1(b)$.

Then there exists a minimal solution x_m of the BVP (2.1)-(2.2) such that

$x_m(t) \le v(t)$ for all $t \in J$.

Proof. The proof is similar to that of Theorem 2.5 and we omit the details.

We shall now prove the uniqueness of solutions of the BVP (2.1)-(2.2).

Theorem 2.7 Suppose that the hypothesis of Theorem 2.4 hold. And if the function $f(t,x,x^1,T^*x)$ on $J \times R \times R \times R$ satisfies the condition that

$$\left| f(t,x,x^1,T^*x) - f(t,y,y^1,T^*y) \right|$$

$$\le N\ Min\left(\frac{|x-y|}{(b-a)+|x-y|}, \frac{|x^1-y^1|}{(b-a)+|x^1-y^1|}, \frac{|T^*x-T^*y|}{(b-a)+|T^*x-T^*y|} \right) \tag{2.9}$$

for some N>0 with $N\dfrac{(b-a)^2}{8}<1$. Then the BVP (2.1)-(2.2) has unique solution on *J*.

Proof. Let BM (J,R) denote the space of all bounded measurable real valued functions defined on J. Define a norm on $BM(J,R)$ by

$$\|x\|_2 = \frac{8}{(b-a)^2} \sup_{t \in J} |x(t)| \quad \text{for } x \in BM(J,R).$$

Notice that BM(J,R) is a Banach space with the norm $\|.\|_2$. Define $Q: BM(J,R) \to BM(J,R)$ as given in (2.5).

Let $x_M(t), x_m(t)$ be maximal and minimal solutions of the BVP (2.1)-(2.2) respectively .Then we have

$$|Qx_M(t) - Qx_m(t)| \le \int_a^b G(t,s) \left| f(s, x_M(s), x_M^1(s), T^* x_M(s)) - f(s, x_m(s), x_m^1(s), T^* x_m(s)) \right| ds$$

Using (2.9) ,we see that

$$|Qx_M(t) - Qx_m(t)| \le N \int_a^b G(t,s) \left(\frac{|x_M(s) - x_m(s)|}{(b-a) + |x_M(s) - x_m(s)|} \right) ds$$

Thus

$$\frac{8}{(b-a)^2} \sup_{t \in J} |Qx_M(t) - Qx_m(t)| \le N \ \|x_M - x_m\|_2 \ \frac{(b-a)^2}{8}$$

That is

$$\|Qx_M - Qx_m\|_2 \le N \ \frac{(b-a)^2}{8} \|x_M - x_m\|_2 \quad .$$

Since $N \ \frac{(b-a)^2}{8}$<1, the mapping Q is a contraction. Hence by the contraction mapping principle, we conclude that there exists a unique fixed point $x \in BM(J,R)$ such that

$$Qx(t) = x(t) = \int_a^b G(t,s) f(s, x(s), x^1(s), T^* x(s)) ds \quad , t \in J.$$

This completes the proof.

3. Conclusion

The purpose of the present paper is to study existence of extremal solutions of two point boundary value problems ,with out assuming the upper and lower solutions, also see that these maximal, minimal solutions serve as the bounds for the solutions of differential inequalities related to the BVP's. And establish uniqueness of solutions by using fixed point method. The fixed point technique is one of the useful methods largely applied in the existence and uniqueness properties of Differential Equations.

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