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How Difficult To Compute Coefficients of Characteristic Polynomial?

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Abstract: *This article presents a summarization on computing coefficients of characteristic polynomial of a square matrix in point of view of time complexity. Major classical approaches that are reported in computation of determinants and coefficients of the characteristic polynomial are overviewed. Time-complexity and trait of computation are evaluated for each approach.*

Keywords: *Determinant, Characteristic polynomial, Computation, Time complexity*.

1. INTRODUCTION

A recent research pushed me to an algorithm to compute the power of the trace of a square matrix. Then I have to look into various literatures related with the issue. When I read ZHANG's paper [1], I was shocked by her assertion that coefficients of a characteristic polynomial can be easily computed by means of computing an adjoint matrix. Then my interesting was led to see how easy the coefficients of characteristic polynomials are computed. I then found and read all the articles that are list in the bibliographies [2] to [18]. Unfortunately, I could not find an easy approach as [1] declared. On the contrary, I think the computation is quite difficult. Thus in this article I present my acquaintance as well as a survey on the computation of the coefficients of characteristic polynomials.

2. BASIC DEFINITIONS AND OBJECTIVE

Determinant of an $n\times n$ matrix $A=[a_{(i,j)}]$ is defined to be the scalar

$$\det(A)=|A|=\sum_{p}\sigma(p)\prod_{i=1}^{n}a_{(i,p_i)} \tag{1}$$

where the sum is taken over the n! Permutations p = (p_1, p_2, . . . , p_n) of (1, 2, . . . , n).

The characteristic polynomial of A, $f_A(\lambda)$, is defined by

$$f_A(\lambda)=|\lambda I-A|=\lambda^n+c_1\lambda^{n-1}+...+c_k\lambda^{n-k}+...+c_{n-1}\lambda+c_n \tag{2}$$

where the coefficients $c_k\,(k=1,2,...,n)$ are real numbers.

Our objective is to find a way to compute $c_k\,(k=1,2,...,n)$ and to make it clear how difficult the computation is.

Note that $f_A(\lambda)$ is a determinant and from schoolbook of linear algebra [11] it is known

$$c_k=(-1)^k\sum S_k \tag{3}$$

where S_k is A's principal minor of order k.

And particularly,

$$c_n=(-1)^n\,|A| \tag{4}$$

Since S_k and $|A|$ are all determinants, we know that the difficulty of computing the coefficients $c_k\,(k=1,2,...,n)$ is the same as that to compute a determinant if we directly compute $c_k\,(k=1,2,...,n)$

by (3). Thus it is mandatory to know the difficulty of computing a determinant first and then to see the other approaches and their difficulties to compute $c_k\,(k=1,2,...,n)$.

3. COMPLEXITY OF COMPUTING A DETERMINANT

Looking from schoolbooks to professional articles, one can see that algorithms for computing determinants can be generally classified into two kinds: for a general form and for a special form. For example, bibliographies [19] – [25] all concern with determinants of special forms. The foregone literatures show that, computing determinants of special forms is a little easier than that of the general forms. Since the principal minors in (3) are of general forms, we only focus on this sort. Looking through schoolbooks and professional articles on computations of general determinants, we can find three kinds of computations as summarized below.

3.1 Computation Via Basic Properties

Except for basic definition (1), every schoolbook also introduces Laplace expansion as is given by (5)

$$\det(A)=\sum_{i=1}^{n}(-1)^{i+j}a_{(i,j)}M_{(i,j)} \tag{5}$$

where $M_{(i,j)}$ is the algebraic cofactor of? $a_{(i,j)}$

These two kinds of computations are regarded to be original and basic. Steven S. Skiena points out in his book [28] that, both the approaches' time complexity are $n!$, which is very difficult to compute when n is large.

3.2 Computation Via Gaussian Elimination

Gaussian elimination and LU decomposition that are widely used to solve linear equations are also introduced to compute the determinants in schoolbooks as seen in [27] and [28] respectively. These two approaches have an $O(n^3)$ time complexity, as shown in [28].

3.3 Computation By Special Means

Due to different demands in computation from different applicable occasions, development of new methods to compute determinants has never stopped, as stated in Ershaidat's introductory essay [29]. Computation by special means has been an important topic in algorithm design. Among various algorithms, the division-free approach that was early proposed by Kaltofen [30] and parallel approaches are particularly concerned. Gunter Rote and T R Seifullin proposed algorithms of $O(n^4)$ and $O(n^{3+0.5})$ respectively in [31] and [32]. According to Villard's summarization [33], the best result of this issue is $O(n^{2.698})$. In 2014, Almalki and his workmates designed parallel algorithms for Laplace expansion and LU decomposition. The algorithms are 129% and 44% respectively faster than the sequential algorithms [34].

4. COMPUTATION OF COEFFICIENTS OF CHARACTERISTIC OF POLYNOMIAL

Now that we know the complexity of computing a determinant, we begin to investigate the computation of the coefficients $c_k\,(k=1,2,...,n)$ in (2). First, it should point out that, it is very difficult to perform the computation directly by (3) because an $n\times n$ matrix has 2^n-1 principal minors in all. Therefore, people have continuously found other approaches.

4.1 Leverrier-Faddeev Algorithm

Leverrier-Faddeev Algorithm is a classical algorithm that is investigated in many articles and frequently introduced in books of linear algebra and linear systems, as seen in the bibliographies [2], [10], [35] and [36]. By the algorithm, coefficients of characteristic polynomial (2) are computed by the following recursive procedure.

$$\begin{aligned}
&B_1=I,B_k=AB_{k-1}+c_{k-1}I,k=2,...,n\\
&0=AB_n+c_nI\\
&c_k=-\frac{1}{k}Tr(AB_k),k=1,2,...,n
\end{aligned} \tag{7}$$

According to J C Gower.'s article, the original Leverrier- Faddeev algorithm takes an $O(n^4)$ time complexity and it is not very good for numerical work [35]. Therefore, there appeared modified versions of the algorithm, as seen in [37] to [39]. However, the modifications do not change the time complexity.

4.2 Silva's Formula

R R Silva in 1998 put forward a recursive formula in [8] and [9] that can express coefficients of characteristic polynomial by traces of matrix. H H ZHANG and his workmates obtained the same result in 2008 [17]. The formula is given by

$$T_k + c_1 T_{k-1} + c_2 T_{k-2} + \ldots + c_i T_{k-i} + \ldots + c_{k-1} T_1 + k c_k = 0, k = 1, 2, \ldots, n \quad (6)$$

Where $T_k = \sum_i \lambda_i^k$, and $\lambda_i (i = 1, 2, \ldots, n)$ are complex eigenvalues of the matrix A.

Neither [8] nor [9] evaluate the time complexity of the algorithm. But we can guess that it is at least $O(n^3)$ because it requires $O(n^3)$ to obtain all the n eigenvalues (e.g., by QR decomposition).

4.3 Algorithms Via Matrix Transformation

According to matrix theory, two similar matrices, A and B, have the same characteristic polynomial. This fact leads to algorithms to obtain A's characteristic polynomial via B's if the B's is easier to compute. Danilevskii algorithm, Hessenberg Algorithm and Frobenius Algorithm are such kind of algorithms, as summarized in [40] and [41]. These algorithms are of $O(n^3)$ time complexity and each algorithm has its shortcoming in view of computation, as pointed out in [41].

4.4 Other Formulas

There are several other formulas to compute coefficients of characteristic polynomial, such as in [3], [4], [5], [7], [13], [14], [15] and [17]. Looking into these articles, one can see that these formulas just give an equivalent expression of the coefficients in (2); they do not reduce the time complexity, some even increase the time complexity as the articles [15] and [17] do. For example, the article [15] gives a formula by

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ \ldots \\ c_{n-1} \\ c_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}T_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3}T_2 & \frac{1}{3}T_1 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4}T_3 & \frac{1}{4}T_2 & \frac{1}{4}T_1 & 1 & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{1}{n-1}T_{n-2} & \frac{1}{n-1}T_{n-3} & \frac{1}{n-1}T_{n-4} & \ldots & \frac{1}{n-1}T_1 & 1 & 0 \\ \frac{1}{n}T_{n-1} & \frac{1}{n}T_{n-2} & \frac{1}{n}T_{n-3} & \ldots & \frac{1}{n}T_2 & \frac{1}{n}T_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -T_1 \\ -\frac{1}{2}T_2 \\ -\frac{1}{3}T_3 \\ \frac{1}{4}T_4 \\ \ldots \\ -\frac{1}{n-1}T_{n-1} \\ -\frac{1}{n}T_n \end{pmatrix}$$

Where $T_k = Tr(A^k)(k = 1, 2, \ldots, n)$ is the trace of matrix A^k.

This formula is a time consuming one because it requires an inverse matrix and it will cost a lot of time to compute T_k by $Tr(A^k)(k = 1, 2, \ldots, n)$ when k is large. So I think the formula is trivial. In the article [17], coefficient c_k is given by

$$c_k = \sum_{(p_1, p_2, \ldots, p_k) \in S_k} \prod_{m=1}^{k} \frac{1}{p_m!} \left[\frac{T_m}{m} \right]^{p_m}$$

Where $T_m = Tr(A^m)$ and S_k is a set including all non-negative integer solutions $\{(p_1, p_2, \ldots, p_k)\}$ that fits $\sum_{i=1}^{k} i p_i = k$.

It is obvious that the formula increases a lot of accessory computations and it is also trivial.

5. CONCLUSIONS

Computation of a characteristic polynomial is a classical problem in both scientific research and technical development. The various algorithms that are designed by researchers from different areas show that the problem is never an easy one. The time complexity of the computation may take from an $O(n^3)$ to $O(2^n)$ and fast algorithm is still in needs. This leaves us a research space that needs rigorous and hard working.

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