International Journal of Research Studies in Computer Science and Engineering (IJRSCSE)
Volume 3, Issue 1, January 2016, PP 1-6
ISSN 2349-4840 (Print) & ISSN 2349-4859 (Online)
www.arcjournals.org

On The Illumination of Polygons by 60°-Floodlights

Marzieh Eskandari[1], Atieh Darvishvand[2]
Faculty of Mathematical Sciences, Alzahra University,Tehran, Iran.
[1] *eskandari@alzahra.ac.ir*, [2] *darvishvand93@gmail.com*

Abstract: *In this paper, first we consider the problem of finding the minimum number of floodlights that can illuminate the interior of an orthogonal polygon when the range of illumination is restricted to 60°. Then we study this problem for the pseudo-triangles. In general, we want to guard an environment with cameras in order to ensure that every point in the environment is seen from at least one camera. We intend to minimize the total number of cameras required and also the cameras can see a limited range. This problem is answered before for some range of visions but there was only a conjecture for 60°-cameras [6]. We prove the correctness of this conjecture for the orthogonal polygons and the pseudo-triangles.*

Keywords: *Art Gallery; Orthogonal Polygons; Visibility; Computational Geometry; Pseudo-triangles.*

1. Introduction

Consider the problem of guarding an environment with cameras in order to ensure that every point in the environment is seen from at least one camera [3]. We intend to minimize the total number of cameras required. This is known as the art gallery problem.

In other words, we want to illuminate a polygonal environment by minimum number of floodlights. Consider the problem of illuminating the interior of P with floodlights which can illuminate a certain portion of plane called α-floodlight. We intend to minimize the total number of α-floodlights required

Definition 1. An α-floodlight, $\alpha \in (0°, 360°]$, is a pair (p, C_α) of a point p and a cone C_α of aperture α at apex p. We say that a point $q \in P$ is illuminated by p if $pq \subset P \cap C_\alpha$.

In classical art gallery problem, a floodlight can illuminate whole around itself, i.e., $\alpha = 360^\circ$. This problem posed by V.Klee in 1978[2]. Then V.Chvatal [1] established that $\left\lfloor \frac{n}{3} \right\rfloor$ guards are always sufficient and sometimes necessary for guarding an n-sided simple polygon.

The aim is to place the least number of floodlights inside a polygon to illuminate interior of the polygon. Variety of angles of illumination causes different cases of this problem. Up to now, this problem was discussed for$\alpha = 360^\circ$,$\alpha \in [180°, 360°)$, $\alpha \in [90°, 180°)$ and $\alpha \in [45°, 60°)$.

Then Urrutia asked a question: "What is the minimum number of guards ($f(n)$) for full protection of each n-gon while angle of vision of guards are $180°$ and they can locate on each point?" [6]. An obvious upper bound is $2\left\lfloor \frac{n}{3} \right\rfloor$, by replacing every $360°$-floodlight by two $180°$-floodlights. In 1992, Larman and Bunting showed $f(n) \leq \left\lfloor \frac{4}{9}\left(n + \frac{1}{4}\right) \right\rfloor$ and this result was improved to $f(n) \leq \left\lfloor \frac{2}{5}(n-3) \right\rfloor$ by Csizmadia and Toth. Finally, $f(n) \leq \left\lfloor \frac{n}{3} \right\rfloor$ was presented as an upper bound for the maximum number of guards [4]. In 2002, Thoth proved that for $\alpha \in [90°, 180°)$, the minimum number of floodlights is $2\left\lfloor\frac{n}{3}\right\rfloor$[5] and then in 2003, he proved that for $\alpha \in [45°, 60°)$, $f(n) \leq n-1$ for odd values of n and $f(n) \leq n-2$ for even values of n [6]. For $\alpha \in [60°, 90°)$, the problem is still open.

Let $f(n,\alpha)$ be as the minimum number of α-floodlights which can cover an arbitrary polygon, as defined in [6]. Note that the function $g(\alpha) = \lim_{n\to\infty} Sup \frac{f(n,\alpha)}{n}$ is a monotone function and $\lim_{\alpha\to 0°} g(\alpha) = \infty$ [6]. Figure 1 shows the known values of this function.

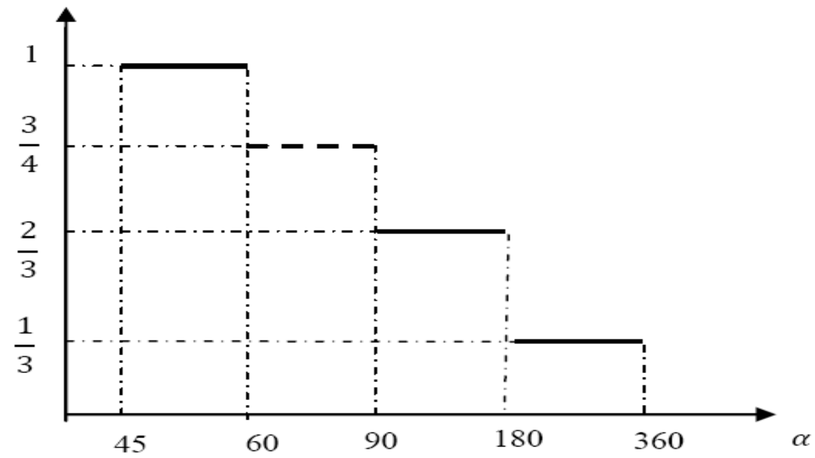

Fig1. *Representation of G(A)*

According to the definition of g(α) and its graph, Toth expected that $f(n, 60°) = \left\lceil\frac{3(n-2)}{4}\right\rceil$. The lower bound $\left\lceil\frac{3(n-2)}{4}\right\rceil$ has been proved by Toth [6]. For the upper bound, there is no result, up to now.

2. Orthogonal Polygons

In this section we consider the orthogonal polygons. In an orthogonal polygon every edge is either horizontal or vertical, and thus every vertex is a right angle. Let P be a simple orthogonal polygon without hole which has n vertices. Consider the problem of illuminating the interior of an orthogonal polygon P with minimum number of 60°-floodlights.

As we mentioned before, there is a lower bound for the number of required 60°-floodlights to illuminate an arbitrary polygon [6]. In this section, we will show that this lower bound is also an upper bound for the orthogonal polygons. Precisely we want to show that $\left\lceil\frac{3(n-2)}{4}\right\rceil$ 60°–floodlights are sufficient for illuminating an orthogonal polygon with n vertices.

There are two cases:

- There is a horizontal or vertical line segment $\boldsymbol{l}$ connecting two reflex vertices of $\boldsymbol{P}$ such that the interior of $\boldsymbol{l}$ is totally contained in the interior of $\boldsymbol{P}$.
- No such horizontal or vertical line segment exists. In this case, we say that $\boldsymbol{P}$ is in general position.

Lemma 1. Let P be an orthogonal polygon with r reflex vertices that satisfies Constraint (I), then it can be illuminated by $2\left\lfloor\frac{r}{2}\right\rfloor + 2$ 60°-floodlights.

Proof. Let l be a horizontal line segment as in (I). Then l splits P into two orthogonal polygons P_1 and P_2 with r_1 and r_2 reflex vertices respectively, such that $r_1 + r_2 = r - 2$. By induction, we can illuminate them with $2\left\lfloor\frac{r_1}{2}\right\rfloor + 2$ and $2\left\lfloor\frac{r_2}{2}\right\rfloor + 2$ 60°-floodlights respectively. But we know that $2\left\lfloor\frac{r_1}{2}\right\rfloor + 2 + 2\left\lfloor\frac{r_2}{2}\right\rfloor + 2 \le 2\left\lfloor\frac{r}{2}\right\rfloor + 2$, so we are done.

Now we will show that the above lemma holds also for the orthogonal polygons in general position. For this we need some notations and use a partition of orthogonal polygons into L-shape [7].

Definition 2. An orthogonal polygon is called L-shape, if it has at most one reflex vertex.

Note that an L-shape polygon is a rectangle or an orthogonal hexagon. See Figure 2.

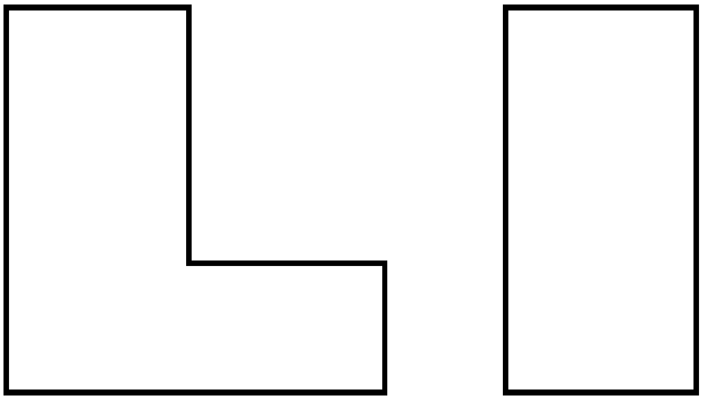

Fig2. *L-shape polygon*

The main idea for proving Lemma 1 for arbitrary orthogonal polygons is the following lemma which is proved in [7].

Lemma 2. Every orthogonal polygon in general position with r reflex vertices can be partitioned into at most $\left\lfloor\frac{r}{2}\right\rfloor + 1$ L-shape polygons.

Proof. Refer to [7].

Lemma 3. Let P be an orthogonal polygon in general position with r reflex vertices. Then it can be illuminated by $2\left\lfloor\frac{r}{2}\right\rfloor + 2$ 60°-floodlights.

Proof. First we partition P into $\left\lfloor\frac{r}{2}\right\rfloor + 1$ L-shapes by Lemma 2. Each L-shape polygon can be illuminated by two 60°-floodlights. See Figure 2. Then we can illuminate P by $2\left\lfloor\frac{r}{2}\right\rfloor + 2$ 60°-floodlights.

Obsevation 1. The number of reflex vertices of an $\boldsymbol{n}$-gon $\boldsymbol{P}$ is $\frac{(\boldsymbol{n}-\mathbf{2})}{\mathbf{4}}$.

Theorem 1. Let P be an orthogonal polygon with n vertices, then it can be illuminated by $\left\lceil\frac{3(n-2)}{4}\right\rceil$ 60°-floodlights.

Proof. By Lemmas 1 and 3, we can illuminate P by $2\left\lfloor\frac{r}{2}\right\rfloor + 2$ $60°$-floodlights which equals to $2\left(\left\lfloor\frac{n-4}{4}\right\rfloor + 1\right)$ by Observation 1. Note that n is even, therefore always we have $2\left\lfloor\frac{n}{4}\right\rfloor \leq \left\lceil\frac{3(n-2)}{4}\right\rceil$. So the number of floodlights is at most $\left\lceil\frac{3(n-2)}{4}\right\rceil$.

3. Pseudo-Triangles

In this section, we study the problem of illuminating pseudo-triangles by $\mathbf{60°}$–floodlights and we will show that the lower bound $\left\lceil\frac{\mathbf{3(n-2)}}{\mathbf{4}}\right\rceil$ is also an upper bound for this class of polygons. In other words, we will show that $\left\lceil\frac{\mathbf{3(n-2)}}{\mathbf{4}}\right\rceil$ $\mathbf{60°}$–floodlights are sufficient for illuminating a pseudo-triangle with $\boldsymbol{n}$ vertices.

A pseudo-triangle is the “most reflex” polygon possible—it has exactly three convex vertices with internal angles less than $\mathbf{180°}$. See Figure 3.

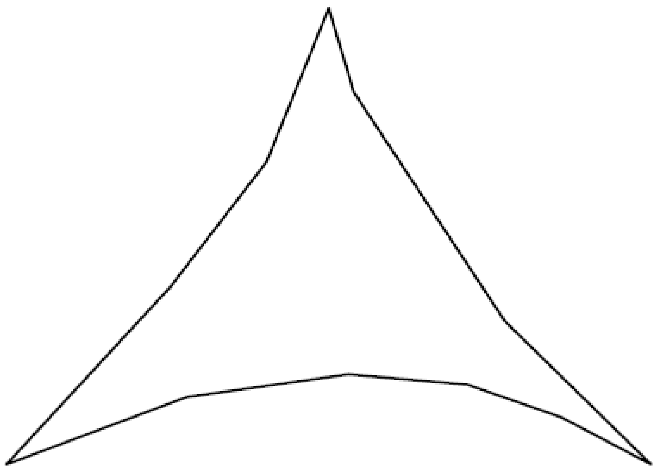

Fig3. *A pseudo-triangle*

Clearly a pseudo-triangle is a star shaped polygon, it means that there is point $\boldsymbol{S}$ inside it which is visible from all points of pseudo-triangle. If we place three $\mathbf{60°}$-floodlights on $\boldsymbol{S}$, the pseudo-triangle can be illuminated. Therefore each pseudo-triangle can be illuminated by six floodlights. If $\mathbf{6} \leq \left\lceil\frac{\mathbf{3(n-2)}}{\mathbf{4}}\right\rceil$, then we are done. In other words, for $\boldsymbol{n} \geq \mathbf{9}$, $\left\lceil\frac{\mathbf{3(n-2)}}{\mathbf{4}}\right\rceil$ is an upper bound.

Now we assume that $\mathbf{3} \leq \boldsymbol{n} \leq \mathbf{8}$. We should prove that $\left\lceil\frac{\mathbf{3(n-2)}}{\mathbf{4}}\right\rceil$ $\mathbf{60°}$–floodlights are sufficient for illuminating these eight classes of pseudo-triangles with $\boldsymbol{n}$ vertices.

1. $\boldsymbol{n} = \mathbf{3}$: each triangle has at least one vertex less that $\mathbf{60°}$. So it can be illuminated with one $\mathbf{60°}$–floodlight.
2. $\boldsymbol{n} = \mathbf{4}$: these pseudo-triangles are tetragons with a reflex vertex, which can be partitioned into two triangles. So it can be illuminated with two $\mathbf{60°}$–floodlights.
3. $\boldsymbol{n} = \mathbf{5}$: always we can partition these pentagons into a triangle and a tetragon**.** So it can be illuminated with three $\mathbf{60°}$–floodlights, one for the triangle and two for the tetragon.
4. $\boldsymbol{n} = \mathbf{6}$: Figure 4 shows all the cases of hexagons which are also pseudo-triangle.

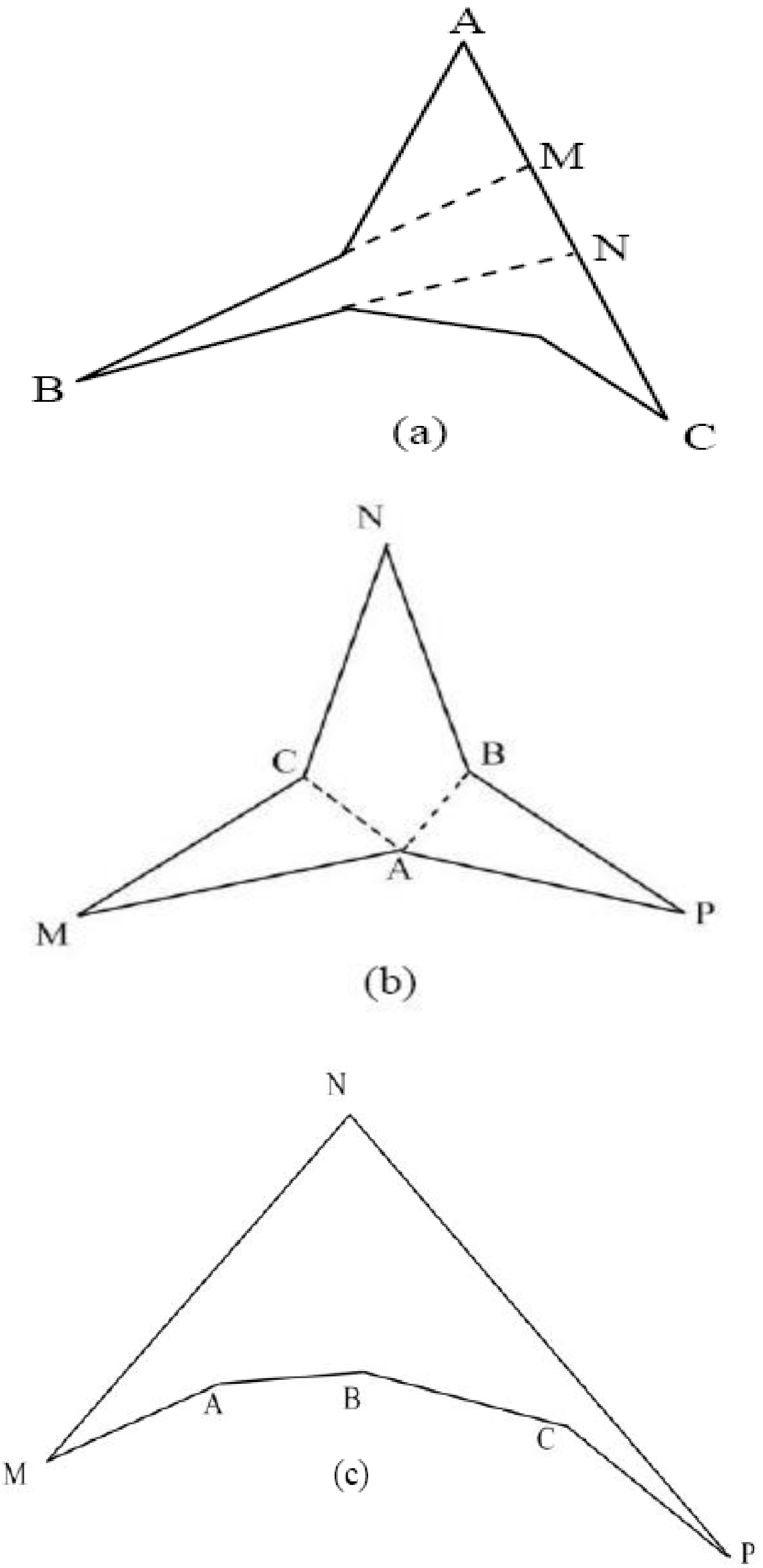

Fig4. *All types of hexagons*

In case (a), we place three $\mathbf{60}$°–floodlights on a point on segment MN.

In case (b), clearly one of the convex vertices is less than $\mathbf{60}$°, denoted as N. We can partition the hexagon into a tetragon with a vertex which is less than $\mathbf{60}$° and two triangles. See Figure 4(b). The tetragon needs one floodlight. So the hexagon can be illuminated by three floodlights.

In case (c) we place three floodlights on convex vertex N. See Figure 4(c).

5. $\boldsymbol{n = 7}$: always we can partition these polygons into a triangle and a hexagon. So it can be illuminated with four $\mathbf{60}$°–floodlights, one for the triangle and three for the tetragon.

6. $\boldsymbol{n = 8}$: always we can partition these polygons into a triangle and a 7-gon. So it can be illuminated with five $\mathbf{60}$°–floodlights, one for the triangle and four for the tetragon.

4. CONCLUSIONS AND FUTURE WORKS

In this study, we considered the problem of illuminating the interior of P with minimum number of 60°-floodlights, where the polygons can be orthogonal or pseudo-triangle. We proved the correctness of the conjecture of $\left\lceil \frac{3(n-2)}{4} \right\rceil$ for the orthogonal polygons and the pseudo-triangles.

REFERENCES

[1] V. Chva´tal. A combinatorial theorem in plane geometry. Journal of Combinatorial Theory Ser B 18, pp. 39-41, (1975).

[2] V. Klee. Is every polygonal region illuminated from some point?. Amer. Math. Monthly 76, p. 180, (1969).

[3] J. O'Rourke, Art gallery theorems and algorithms. Oxford University Press Oxford, 1987, vol. 1092.

[4] Cs.D. Toth, Art gallery problem with guards whose range of vision is 180, Comput. Geom. Theory Appl. 17 (2000) 121–134.

[5] Cs.D. Toth, Art galleries with guards of uniform range of vision, Comput. Geom. Theory Appl. 21 (2002) 185–192.

[6] Cs.D. Toth, Illumination of polygons by 45-floodlights, Discreet Mathematics, 265 (2003) 251–260.

[7] J. Urrutia, Art gallery and illumination problems, in: J.R. Sack, J. Urrutia (Eds.), Handbook on Computational Geometry, Elsevier, Amsterdam, 2000, pp. 973–1027.

AUTHORS' BIOGRAPHY

Marzieh Eskandari, is an assistant professor in computer sciences at Alzahra University. She is also a member of algorithm and Computational geometry research Group, ACG laboratory, at Amirkabir University of technology. In October 2009, she completed her Ph.D. In applied mathematics (computational geometry and algorithms) at amirkabir university of technology, where she also got her M.Sc. in October 2001 in the same field and her B.Sc. is from Sharif University of technology in applied mathematics.

Atieh Darvishvand, received the B.Sc. in applied mathematics from alzahra University, Iran (2012). She also received M.Sc. in applied mathematics (computational geometry and algorithms) from alzahra University, Iran (2014). She was awarded for the best graduate student in B.Sc. Her research interests include, Computational Geometry, Visibility, and Art Gallery.