International Journal of Modern Studies in Mechanical Engineering (IJMSME)
Volume 4, Issue 4, 2018, PP 39-46
ISSN 2454-9711 (Online)
DOI: http://dx.doi.org/10.20431/2454-9711.0404005
www.arcjournals.org

Modification of the Classical Theory of the Virtual Mass of an Accelerating Sphere in Fluids

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Abstract: *A semi-analytical solution to the virtual mass of an accelerating spherical particle is obtained. Boundary layer separation coupled with potential flow was assumed around the solid sphere. The new solution of the virtual mass coefficient C_m converges to the original solution of 0.5 upon removing the separation. The solution is valid in the range of Reynolds number from 10 to 1000. The solution compared well with the experimental results of other investigators.*

Keywords: *Fluid Mechanics; Dynamics of Rigid Systems; Virtual Mass; Added Mass; Spherical Particle; Boundary Layer*

1. Introduction

When a solid sphere accelerates in a fluid, a force, which is not present in steady flow conditions, must be accounted for in the general equation of the conservation of momentum. This force is called the virtual or added mass force. It arises from accelerating the fluid around the sphere.

The drag force and buoyancy for the submerged sphere are not sufficient. In addition, the fluid has the same effect as increasing the mass of the sphere by one-half the mass of the displaced fluid.

The virtual mass of the sphere derived earlier (Milne-Thomson [1], p.491)was based on potential flow. One may note that in these derivation the streamline flow persists around the entire surface of the sphere. It can be concluded that the one- half virtual mass coefficient is not always constant, as the author showed that the virtual mass coefficient of the rotating sphere is equal to five [2].

The importance of the virtual mass concept was appreciated by Lahey et al. [3] who demonstrated that the inclusion of the virtual mass effects into the numerical solution of the transient two-phase evapoative flow appears to improve numerical stability and efficiency and to achieve accurate results in many cases of practical concern.

Thorley and Wiggert [4] found that the addition of virtual mass provides more accurate and generalized expressions for the accoustic propagation velocity of two-phase glass bead-water flow.

Magnaudet et al. [5] studied numerically the forces acting on a sphere ebedded in accelerated flows. They concluded that a virtual mass coefficient equal to 0.5 is used for ceeping flow. They demonstrated the existence of the separation angle, θ_S of the boundary layer and they showed that the separated region is much more reduced in accelerated flow than in uniform flow. The influence of the accelerated flow on the evolution of the separation angle was manifested by the shift of the critical Reynolds number, Re_C to a higher than 20. For rigid sphere in uniform flow, it is well known (see e.g. Batchelor [6] that a separated region and eddy first appear at $\mathrm{Re}_C \approx 20$).

Sano [7] solved the unsteady flow past a sphere and found a crtical time for the first appearance of eddy at the wake corresponding to Re_C.

Luneau [8], Batailler [9] and McNown and Keulegan [10] attempted to show experimentally that the virtual mass could be explained physically by the wake of the body being accelerated with it.

There is uncertainty in the reported values of Cm. For example, Iverson and Balent [11] showed that Cm ranges from 0.5 to 2 as reported by many experimental and theoretical investigators.

The aim of the present paper is to derive an equation for the virtual mass coefficient of a solid sphere accelerating at high speed. Boundary layer separation and wake formation are normally associated with high-speed flow.

The flow field around the solid sphere will be divided into distinct regions the forward and the wake regions separated by the separation ring of the boundary layer as shown in Fig.1. This technique have been used before by the author [12] and [13].

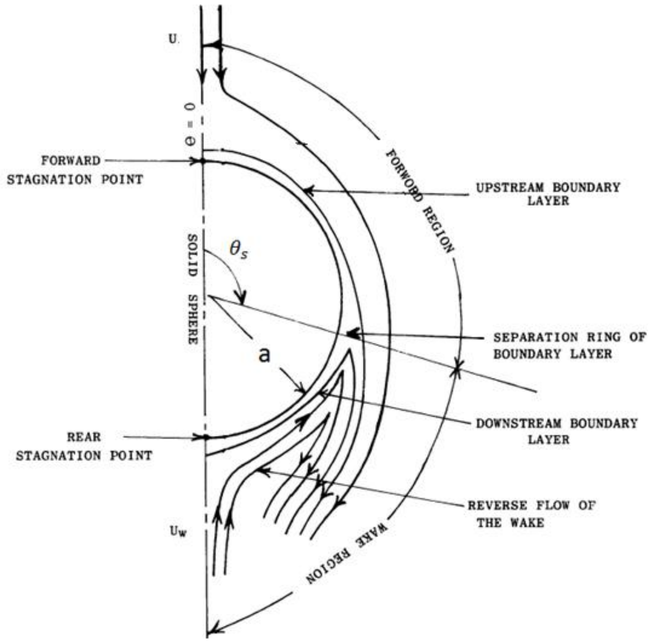

Fig1. *Accelerating sphere in a fluid.*

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2. The Forward Region Of The Sphere

Consider a solid sphere of radius a moving with speed U through an incompressible nonviscous fluid of density ρ. Assume that the fluid is stationary at infinity. The potential ϕ is the dipole potential given in spherical coordinates with the origin at the center of the sphere by the following

$$\phi = \frac{1}{2} U \frac{a^3}{r^2} \cos\theta \quad (1)$$

The velocity field $\bar{u}$ is given by

$$\bar{u} = \frac{\partial \phi}{\partial r} \bar{i}_r + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \bar{i}_\theta \quad (2)$$

The kinetic energy KE1 of the flow around the sphere from the forward stagnation point up to the separation ring is given by

$$KE1 = \int_V \frac{1}{2} \rho |\bar{u}|^2 dV \quad (3)$$

where V denotes the volume occupied by the sphere. The volume dV of an elementary annular region about the "$\theta = 0$" line through the sphere is given by the product of its perimeter $(2\pi r \sin\theta)$, width $(rd\theta)$ and depth (dr), Thus

$$KE1 = \int_{r=a}^{\infty} \int_{\theta=0}^{\theta_s} \frac{1}{2} \rho ({u_r}^2 + {u_\theta}^2) 2\pi r^2 \sin\theta dr d\theta \quad (4)$$

The upper limit of the second integral is the angle of separation that will be denoted as $\theta_s(r,t)$ as it varies spatially and temporally in the accelerated flow. Substituting Eq.(2) and performing the integration, we get

$$KE1 = -\frac{\pi}{12}\rho U^2 a^3 (cos\theta_s(r,t) + cos^3\theta_s(r,t) - 2) \tag{5}$$

As $\theta_s(r,t) \rightarrow \pi$, we recover the expression given by Milne-Thomson [1] (page 491) for the kinetic energy of the complete sphere, that is $KE = \frac{1}{3}\pi\rho U^2 a^3$.

The above method of solution was adopted by the author [14] for the derivation of the virtual mass of the spherical-cap bubble. The forward region of this bubble is always spherical with an almost flat base.

3. The Wake Region Of The Sphere

Consider the reverse flow in the wake region of the sphere to be similar in pattern and streamline to that around the forward region of a stagnant sphere in a flow, particularly in the vicinity of the surface.This assumption is not far from reality if we examine the photographs of the streamlines of the flow past the sphere taken by Taneda [15] and shown in Batchelor [6], plate 3.

The method of the solution used here starts from the rear stagnation point and proceeds backward to the stagnation ring. The above assumption means that $\theta = 0$ will be at the rear stagnation point, and the origin of the r, θ coordinates is still at the center of the sphere.

Now we analyze the wake of the sphere with a main reverse flow velocity U_W (Fig. 1), hence the velocity potential becomes (Milne- Thomson [1], P.488)

$$\phi_W = U_W \cos\theta(-r + \frac{1}{2}\frac{a^3}{r^2}) \tag{6}$$

According to the above assumption, we get the following equation for the kinetic energy of the fluid in the wake region (Lamb[16], p.123)

$$KE2 = -\frac{1}{2}\rho \iint (\phi_W \frac{\partial \phi_W}{\partial r})_{r=a} dA \tag{7}$$

where dA is the differential area of the sphere surface facing the wake, and

$$dA = 2\pi a^2 \sin\theta \, d\theta \tag{8}$$

Substituting Eq.(6) , its differential, and Eq.(8) into Eq.(7)yields the following

$$KE2 = -\pi\rho a^3 U_W^2 \int_0^{\theta_s(r,t)} cos^2\theta sin\theta d\theta \tag{9}$$

which becomes the following after the integration

$$KE2 = \frac{1}{3}\pi\rho a^3 U_W^2 (cos^3\theta_s(r,t) - 1) \tag{10}$$

Lee and Barrow [17] found that the ratio $U_W / U = 0.077$ for the Reynolds number $(\mathrm{Re} = 2a\rho U/\mu)$ range of 10-1000. These Re number limits dictated the range of applicability of the present model. The transformation to the original r, θ coordinates requires the replacement of every $\theta_s(r,t)$ in Eq.(10) by $(\pi - \theta_s(r,t))$,so that Eq.(10) becomes

$$KE2 = -\frac{1}{3}\pi\rho a^3 (0.077U)^2 (cos^3\theta_s(r,t) + 1) \tag{11}$$

The total kinetic energy (KE) of the fluid around the sphere would be sum of the forward and the wake contributions, thus

$$KE = KE1 + KE2 \tag{12}$$

or by substituting Eqs.(5) and 11 into Eq.(12), we get

$$KE = \frac{1}{2}\pi a^3 \rho U^2 \left\{ -\frac{1}{6}(cos\theta_s(r,t) + cos^3\theta_s(r,t) - 2) - \frac{2}{3}(0.005929)(cos^3\theta_s(r,t) + 1) \right\} \tag{13}$$

If we now use the classical form of the kinetic energy as

$$KE = \frac{1}{2} MU^2 \tag{14}$$

where M denotes the mass of fluid moving with a uniform speed U which has the same kinetic energy as the unbounded fluid around the sphere.Clearly, from Eqs.(13) and (14) we get

$$M = -\pi\rho a^3\{(cos\theta_s(r,t) + cos^3\theta_s(r,t) - 2) - (0.003953)(cos^3\theta_s(r,t) + 1)\} \tag{15}$$

This equation represents the "virtual mass" of the sphere . The virtual mass coefficient could be defined according to Cheng, Drew and Lahey [18] by the following

$$C_m = \frac{\text{Volume of "virtual mass"}}{\text{Volume of fluid displaced by the sphere}} \tag{16}$$

Applying Eq.(15) into the above ratio, we get

$$C_m = \frac{-\pi a^3\left\{\frac{1}{6}(cos\,\theta_s(r,t) + cos^3\theta_s(r,t) - 2) + (0.003953)(cos^3\theta_s(r,t) + 1)\right\}}{\frac{4}{3}\pi a^3}$$

or the following simplified form

$$C_m = -\frac{3}{4}\left\{\frac{1}{6}(cos\theta_s(r,t) + cos^3\theta_s(r,t) - 2) + (0.003953)(cos^3\theta_s(r,t) + 1)\right\} \tag{17}$$

as $\theta_s \to \pi$ we recover the classical value of C_m= 1/2 for the accelerating sphere at low Reynolds number with no boundary layer separation. However, the wake contribution in Eq.(17) is too small in comparison to the forward portion. Figure 2 shows the variation of C_m with respect to $\theta_s(r,t)$ where C_m approahes the value half at $\theta_s(r,t) = \pi$.

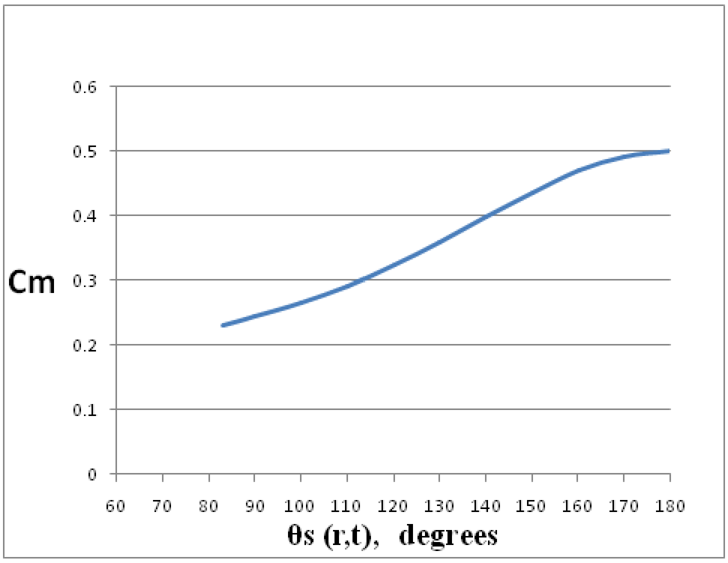

Fig2. *The variation of the virtual mass coefficient with the separation angle.*

The separation angle can be obtained from the correlations of Linton and Sutherlands [19] as follows

$$\theta_s = 83 + 660\,\mathrm{Re}^{-1/2} \qquad \text{For Re>100} \tag{18}$$

and

$$\theta_s = 83 + 191\,\mathrm{Re}^{-1/3} \qquad \text{For 15< Re<1000} \tag{19}$$

Both expression are asymptotic to $\theta_s = 83^\circ$ for large Re number. Accordingly, Eq.(17) becomes the following

$$C_m = 0.2316 \qquad \text{For } \theta_s = 83^\circ \tag{20}$$

The variation of Cm with Re is shown in Fig. 3 which was based on Eqs. (17), (18),(19), and (20). At low Re number flow, the value of Cm approaches the classical value of half.

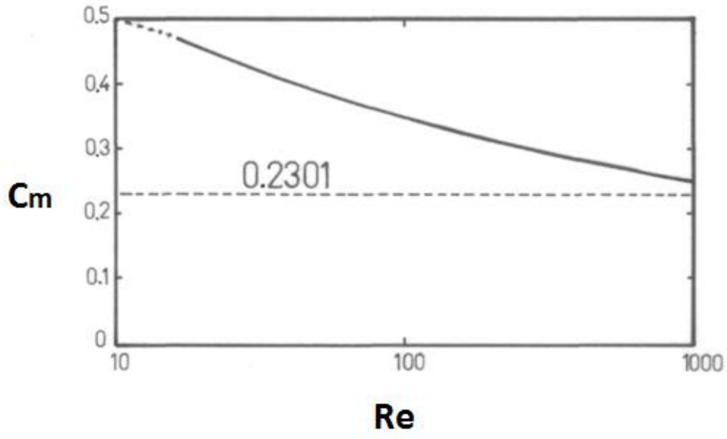

Fig3. *The variation of the virtual mass coefficient with Reynolds number*

4. Discussion

It should be pointed out that the use of potential flow is quite legitimate in virtual mass formulation, but the use of boundary layer flow is the novelty of this work. To further validate the present results, a comparison was made with the experimental data of Moorman [20] where the agreement is fair as shown in Fig. 4.

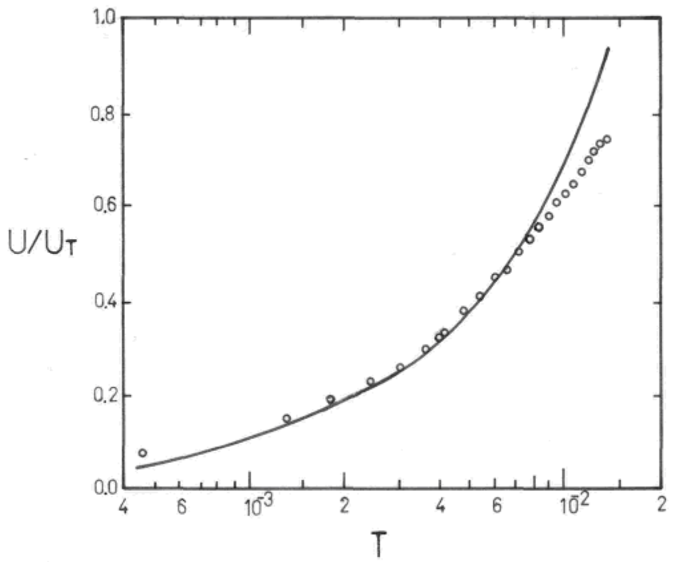

Fig4. *Comparison of the present analytical solution (Eqs. (23) and (24)) with the experimental data of Moorman [20].* $\rho_o = 7782$ *kg/m*^{3}, ρ *= 876.1kg/m*^{3}, ν *=1.0776x10*$^{-5}$*m*^{2}*/s,* a *= 6.355x10*$^{-3}$ *m.*

The following equation of motion was used to calculate the instantaneous velocity of the solid sphere

$$\frac{d}{dt}\left[(4/3)\pi a^3(\rho_o + C_m\rho)U\right] = (4/3)\pi a^3(\rho_o - \rho)g - D \tag{21}$$

The first term on the right side of this equation represents the buoyancy and gravity forces. Here *D* is the drag force given as follows (Kendoush [13]).

$$D = \pi a \mu U f(\theta_s) \tag{22}$$

Where

$$f(\theta_s) = 9H_f(\theta_s) + 0.693H_W(\theta_s),$$

$$H_f(\theta_s) = (2/3) + (1/3)\cos^3\theta_s - \cos\theta_s$$

And

$$H_W(\theta_s) = (2/3) - (1/3)\cos^3\theta_s + \cos\theta_s$$

Substituting Eqs.(17) and (22) into Eq.(21) yields a first order differential equation that was solved readily by the integrating factor method to get the solution shown in Fig. 4. The solution of Eq. (21) was done at every time step of run number 1 of Table 1 of Moorman [20]. For the particular case of Re = 62, C_m=0.3652, $\rho_o = 7782$ kg/m^3, ρ = 876.1kg/m^3, ν =1.0776x10^{-5}m^2/s, a = 6.355x10^{-3} m and for the initial conditions of $U = 0$ at $t = 0$, the solution of Eq.(21) is given as follows

$$U = 34.7138\left(1 - e^{-0.24087t}\right) \tag{23}$$

In Fig. 4, U_T is the terminal velocity of the solid sphere (shown by the ratio U/U_T of the ordinate of Fig. 4), and given by the following

$$U_T = (2a)^2 g(\rho_o - \rho)/(18\nu\rho) \tag{24}$$

The parameter T in Fig. 4 is a dimensionless time equals $t\nu/(2a)^2$.

The history term was neglected in Eq. [21] as being small by several orders of magnitude in comparison to the other terms. Raju and Meiburg [21] reached the same conclusion.

The present model is agreable with the following authors who compared well with Moorman's data, as well, as it is shown in Fig. 4 of Chang and Yen [22]:

- Mei and Adrian [23]
- Basset [24]
- Karafilian and Kots [25]
- Odar and Hamilton [26]

The present results contradicts those of Hamilton and Lindell [27] who showed that for spheres falling under gravity the added-mass coefficient was almost always very nearly equal to the value of 0.5 for Reynolds numbers up to 35000.

Takahashi et al. [28] showed a Reynolds number dependence in their experimental investigation of the virtual mass coefficient of an oscillating spherical particle. They produced the following correlation

$$C_m = 0.7 + 15Re^{-0.75} \tag{25}$$

5. CONCLUSIONS

Based on the preceding analysis and discussion, the following conclusions were abstracted:

- The virtual mass coefficient value of 0.5 is to be applied to the solid spherical particle in an impulsive Stokesi an flow.
- Boundary-layer separation has a profound effect on the virtual mass coefficient of the solid spherical particle at high speed accelerated flow.
- The virtual mass coefficient given by Equation 17 is to be applied to the spherical particle in an impulsive high-speed particulate flows.

Notation

A surface area of sphere, m^2

a sphere radius, m

C_V virtual mass coefficient

D drag force, N

KE kinetic energy, J

M mass of fluid displaced by sphere, kg

Re Reynolds number, $(\mathrm{Re} = 2a\rho U/\mu)$

r polar coordinate, m

t time, s

T dimensionless time ($t\nu/(2a)^2$)

U sphere velocity, m/s

$\bar{u}$ flow velocity field, m/s

Greek Symbols

ρ density of fluid, kg/m^3

μ dynamic viscosity of fluid, Ns/m^2

ν kinematic viscosity of fluid, m^2/s

ϕ velocity potential, m^2/s

Subscripts

o solid

r radial direction

s separation of boundary layer

w wake region

θ angular direction

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Citation: *Abdullah Abbas Kendoush, (2018)" Modification of the Classical Theory of the Virtual Mass of an Accelerating Sphere in Fluids"***,** *International Journal of Modern Studies in Mechanical Engineering, 4(4), pp. 39-46. DOI: http://dx.doi. org/10.20431/2454-9711.0404005*