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Solvability Criterion for the Cauchy Problem for the Theory of Elasticity

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Abstract: *In this paper, we consider problem of analytical continuation of a solution of the system equations of the thermoelasticity in bounded domain from its values and values of its strains on a part of the boundary of this domain, i.e., the Cauchy problem and we give a criterion for solvability of the Cauchy problem.*

Keywords: *The Cauchy problem; system theory of elasticity; elliptic system, ill-posed problem, Carleman matrix, regularization.*

1. INTRODUCTION

In this paper, we consider problem of analytical continuation of a solution of the system equations of the thermo elasticity in bounded domain from its values and values of its strains on a part of the boundary of this domain, i.e., the Cauchy problem and we give a criterion for solvability of the Cauchy problem.

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be points of the $3-$dimensional Euclidean space R^3, D be a bounded simply connected domain in R^3 with piecewise-smooth boundary consisting of a piece Σ of the plane $y_3 = 0$ and a smooth surface S lying in the half-space $y_3 > 0$.

Suppose $U(x) = (u_1(x), u_2(x), u_3(x), u_4(x))$ is a vector function satisfies the following system of equations of the thermo elasticity in D [1]:

$$B(\partial_x, \omega)U(x) = 0, \qquad (1.1)$$

where

$$B(\partial_x, \omega) = \left[\!\left[B_{kj}(\partial_x, \omega) \right]\!\right]_{4\times 4}$$

and

$$B_{kj}(\partial_x, \omega) = \delta_{kj}(\mu\Delta + \rho\omega^2) + (\lambda + \mu)\frac{\partial^2}{\partial x_k \partial x_j}, \quad k, j = 1,2,3,$$

$$B_{k4}(\partial_x, \omega) = -\gamma\frac{\partial}{\partial x_k}, \qquad k = 1,2,3,$$

$$B_{4j}(\partial_x, \omega) = -i\omega\eta\frac{\partial}{\partial x_j}, \qquad j = 1,2,3,$$

$$B_{44}(\partial_x, \omega) = \Delta + \frac{i\omega}{\theta},$$

$\delta_{kj} -$ is the Kronecker delta, $i = \sqrt{-1}$, $\omega -$ is the frequency of oscillation and $\lambda, \mu, \rho, \theta$ are its coefficients which characterizing medium, satisfying the conditions

$$\mu > 0, \qquad 3\lambda + 2\mu > 0, \qquad \rho > 0, \qquad \theta > 0, \qquad \frac{\gamma}{\eta} > 0.$$

System (1.1) can be written in the form:

$$\begin{cases} \mu\Delta u + (\lambda+\mu) grad\ div\ u - \gamma\ grad\ v + \rho\omega^2 u = 0 \\ \Delta v + \frac{i\omega}{\theta} v + i\omega\eta\ div\ u = 0\ , \end{cases} \tag{1.2}$$

where $U(x) = (u_1(x), u_2(x), u_3(x), u_4(x)) = (u(x), v(x))$, $u(x) = (u_1(x), u_2(x), u_3(x))$.

Statement of the problem. Let $f = (f_1, f_2, f_3, f_4)^T \in [C^1(S)]^4$, $g = (g_1, g_2, g_3, g_4)^T \in [C(S)]^4$ be given vector-functions. It requires to find (if possible) a vector-function $U(x) \in [C^1(D \cup S) \cap$ $C2(D)4$ such that

$$\begin{cases} B(\partial_x, \omega)U(x) = 0\ \ in\ \ D\ , \\ U(y) = f(y)\ ,\ \ y \in S\ , \\ R\left(\partial_y, n(y)\right) U(y) = g(y), \quad y \in S\ , \end{cases} \tag{1.3}$$

where $R\left(\partial_y, n(y)\right)$ – is the stress operator, i.e.,

$$R\left(\partial_y, n(y)\right) = \left[\!\left[R_{kj}\left(\partial_y, n(y)\right)\right]\!\right]_{4\times4} = \begin{pmatrix} & T & & -\gamma\, n_1 \\ & & & -\gamma\, n_2 \\ & & & -\gamma\, n_3 \\ 0 & 0 & 0 & \frac{\partial}{\partial n} \end{pmatrix},$$

$$T = T\left(\partial_y, n(y)\right) = \left[\!\left[T_{kj}\left(\partial_y, n(y)\right)\right]\!\right]_{3\times3}\ ,$$

$$T_{kj}\left(\partial_y, n(y)\right) = \lambda n_k(y)\frac{\partial}{\partial y_j} + \mu n_j(y)\frac{\partial}{\partial y_k} + (\lambda+\mu)\frac{\partial}{\partial n(y)}\ ,\quad k, j = 1,2,3,$$

$n(x) = \left(n_1(x), n_2(x), n_3(x)\right)$ – is the unit outward normal vector on ∂D at a point y.

Here $[C^k(S)]^4$, $(k = 0, 1, 2, \dots)$ stands for the vector space of all 4-vector valued functions whose components are k times continuously differentiable on a set $D \subset R^3$.

It is known that the system (1.2) is elliptic and problem (1.3) has no more than one solution. However, it is ill-posed, i.e. 1) not for any data there exists a solution; 2) solution do not depend continuously on the Cauchy data on S (see, for example, [2]).Therefore, solvability conditions can not be described in terms of continuous linear functional.

In this paper we will apply the integral representation's method to obtain solvability conditions and a formula for solution of the problem.

2. Construction of the Carleman Matrix and Approximate Solution for the Cap Type Domain for the Cap Type Domain

It is well known that any regular solution $U(x)$ of the system (1.1) is specified by the formula [1]

$$2U(x) = \int\limits_{\partial D} \left(\Psi(x-y,\omega)\left\{R\left(\partial_y, n(y)\right)U(y)\right\} - \right.$$

$$\left. -\left\{\tilde{R}\left(\partial_y, n(y)\right)\tilde{\Psi}(x-y,\omega)\right\}^* U(y)\right) ds_y\ ,\quad x \in D\ , \tag{2.1}$$

where the symbol $\{\cdot\}^*$ – means the operation of transposition, $\Psi(x-y,\omega)$ is the matrix of the fundamental solutions for the system of equations of steady-state oscillations of the thermoelasticity: given by

$$\Psi(x,\omega) = \left[\!\left[\Psi_{kj}(x,\omega)\right]\!\right]_{4\times4}\ ,$$

$$\Psi_{kj}(x,\omega) = \sum_{l=1}^{3}\left[(1-\delta_{k4})(1-\delta_{j4})\left(\frac{\delta_{kj}}{2\pi\mu}\delta_{3l} - \alpha_l\frac{\partial^2}{\partial x_k \partial x_j}\right) + \right.$$

$$\left. +\beta_l\left(i\omega\eta(1-\delta_{j4})\delta_{k4}\frac{\partial}{\partial x_j} - \gamma(1-\delta_{k4})\delta_{j4}\frac{\partial}{\partial x_k}\right) + \delta_{k4}\delta_{j4}\gamma_l\right]\frac{exp(i\lambda_l|x|)}{|x|}\ ,$$

Where

$$\alpha_l = \frac{(-1)^l\left(1 - i\omega\theta^{-1}\lambda_l^{-2}\right)(\delta_{1l} + \delta_{2l})}{2\pi(\lambda + 2\mu)(\lambda_2^2 - \lambda_1^2)} - \frac{\delta_{3l}}{2\pi\rho\omega^2} \ , \ l = 1,2,3; \ \ \sum_{l=1}^{3} \alpha_l = 0 \ ,$$

$$\beta_l = \frac{(-1)^l(\delta_{1l} + \delta_{2l})}{2\pi(\lambda + 2\mu)(\lambda_2^2 - \lambda_1^2)} \ , \ l = 1,2,3; \ \ \sum_{l=1}^{3} \beta_l = 0 \ ,$$

$$\gamma_l = \frac{(-1)^l\left(\lambda_l^2 - k_1^2\right)(\delta_{1l} + \delta_{2l})}{2\pi(\lambda_2^2 - \lambda_1^2)} \ , \ l = 1,2,3; \ \ \sum_{l=1}^{3} \gamma_l = 0 \ ,$$

$$k_1^2 = \rho\omega^2\,(\lambda + 2\mu)^{-1} \ , \ \lambda_3^2 = \frac{\rho\omega^2}{\mu}$$

λ_1^2 and λ_2^2 are defined from the equation

$$\lambda_1^2 + \lambda_2^2 = \frac{i\omega}{\theta} + \frac{i\omega\gamma\eta}{\lambda + 2\mu} + k_1^2 \ , \qquad \lambda_1^2\lambda_2^2 = \frac{i\omega}{\theta}k_1^2 \ ,$$

where $\lambda_1^2 \neq \lambda_2^2$,

$$\widetilde{\Psi}(x,\omega) = \left[\!\left[\widetilde{\Psi}_{kj}(x,\omega)\right]\!\right]_{4\times4} , \qquad \widetilde{\Psi}_{kj}(x,\omega) = \Psi_{kj}(-x,\omega)$$

$$\tilde{R}\left(\partial_y, n(y)\right) = \left[\!\left[\tilde{R}_{kj}\left(\partial_y, n(y)\right)\right]\!\right]_{4\times4} = \begin{pmatrix} & -i\omega\eta\, n_1 \\ T & -i\omega\eta\, n_2 \\ & -i\omega\eta\, n_3 \\ 0\ \ 0\ \ 0 & \dfrac{\partial}{\partial n} \end{pmatrix},$$

Definition. By the Carleman matrix of the problem (1.1), (1.3) we mean an 4×4 matrix $\Pi(y,x,\omega,\sigma)$ depending on the two points y,x and a positive numerical number parameter σ satisfying the following two conditions:

1) $\Pi(y,x,\omega,\sigma) = \Psi(x-y,\omega) + G(x-y,\sigma)$,

where the matrix $G(x-y,\sigma)$ satisfies system (1.1) with respect to the variable y on D, and $\Psi(x-y,\omega$ is a matrix of the fundamental solutions of system (1.1);

2) $$\int\limits_{\partial D\setminus S} \left(|\Pi(y,x,\omega,\sigma)| + \left|R\left(\partial_y, n(y)\right)\Pi(y,x,\omega,\sigma)\right|\right) ds_y \leq \varepsilon(\sigma) \,,$$

where $(\sigma) \longrightarrow 0$, as $\sigma \longrightarrow \infty$; here $|\Pi|$ is the Euclidean norm of the matrix $\Pi = \left[\!\left[\Pi_{kj}\right]\!\right]_{4\times4}$ i.e.,

$$|\Pi|^2 = \left(\sum_{k,j=1}^{4} \Pi_{kj}^2\right).$$

In particular,

$$|U|^2 = \left(\sum_{k=1}^{4} U_k^2\right).$$

From the definition of Carleman matrix and from the Greens formulas it follows that

Theorem2.1. Any regular solution $U(x)$ of system (1.1) in the domain D is specified by the formula

$$2U(x) = \int\limits_{\partial D} \left(\Pi(y,x,\omega,\sigma)\left\{R\left(\partial_y, n(y)\right)U(y)\right\} - \right.$$

$$-\left\{\tilde{R}\left(\partial_y, n(y)\right)\widetilde{\Pi}(y,x,\omega,\sigma)\right\}^{*} U(y)\Big) ds_y\,, \quad x \in D\,, \tag{2.2}$$

where $\Pi(y,x,\omega,\sigma)$ is the Carleman matrix and $\widetilde{\Pi}(y,x,\omega,\sigma) = \widetilde{\Psi}(y-x,\omega) + \tilde{G}(y-x,\sigma)$.

Using this matrix, one can easily conclude the estimate stability of solution of the problem (1.1), (1.3) and also indicate effective method decision this problem as in [3], [4].

With a view to construct an approximate solution of the problem (1.1), (1.3) we construct the following matrix:

$$\Pi(y,x,\omega,\sigma) = [\![\Pi_{kj}(y,x,\omega,\sigma)]\!]_{4\times 4}\,,$$

$$\Pi_{kj}(y,x,\omega,\sigma) = \sum_{l=1}^{3}\Bigg[(1-\delta_{k4})(1-\delta_{j4})\left(\frac{\delta_{kj}}{2\pi\mu}\delta_{3l} - \alpha_l\frac{\partial^2}{\partial x_k\partial x_j}\right) +$$

$$+\beta_l\left(i\omega\eta(1-\delta_{j4})\delta_{k4}\frac{\partial}{\partial x_j} - \gamma(1-\delta_{k4})\delta_{j4}\frac{\partial}{\partial x_k}\right) + \delta_{k4}\delta_{j4}\gamma_l\Bigg]\Phi(y,x,\sigma,i\lambda_l)\,,\ \ k,j=1,2,3,$$

$$\Pi_{4j}(y,x,\omega,\sigma) = \sum_{l=1}^{3}\left\{i\beta_l\omega\eta(1-\delta_{j4})\frac{\partial}{\partial x_j} + \delta_{j4}\gamma_l\right\}\Phi(y,x,\sigma,i\lambda_l)\,,\ \ j=1,2,3,$$

$$\Pi_{k4}(y,x,\omega,\sigma) = \sum_{l=1}^{3}\left\{-\beta_l\gamma(1-\delta_{k4})\frac{\partial}{\partial x_k} + \delta_{k4}\gamma_l\right\}\Phi(y,x,\sigma,i\lambda_l)\,,\ \ k=1,2,3,$$

$$\Pi_{44}(y,x,\omega,\sigma) = \textstyle\sum_{l=1}^{3}\gamma_l\Phi(y,x,\sigma,i\lambda_l)\,, \tag{2.3}$$

Where

$$\Phi(y,x,\sigma,\Lambda) = \frac{1}{-2\pi^2 exp\,(\sigma x_3^2)}\int_0^\infty Im\,\frac{exp\,(\sigma\,w^2)}{w-x_3}\frac{\cos(\Lambda u)du}{\sqrt{u^2+\alpha^2}}\,, \tag{2.4}$$

$$w = i\sqrt{u^2+\alpha^2} + y_3\,, \quad \alpha^2 = (y_1-x_1)^2 + (y_2-x_2)^2\,, \ \ \alpha > 0.$$

The following theorem was proved in [5].

Lemma2.1. For function $\Phi(y,x,\sigma,\Lambda)$, the following formula is valid

$$\Phi(y,x,\sigma,i\Lambda) = \frac{\exp(i\Lambda r)}{4\pi r} + \varphi(y,x,\sigma,\Lambda),\ r = |x-y|, \tag{2.5}$$

where $\varphi(y,x,\sigma,\Lambda) -$ is a regular function that is defined for all y and x satisfies the Helmholtz equation: $\Delta(\partial_y)\varphi + \Lambda^2\varphi = 0\,,\ y\in D\,,\ \Lambda^2 > 0.$

Moreover, for function $\Phi(y,x,\sigma,i\Lambda)$ holds following inequality

$$\int_{\partial D\setminus S}\left(|\Phi(y,x,\sigma,i\Lambda)| + \left|\frac{\partial\Phi(y,x,\sigma,i\Lambda)}{\partial n}\right|\right)ds_y \le C(\Lambda,D)\sigma exp(-\sigma x_3^2)\,, \tag{2.6}$$

Where $C(\Lambda,D)$ certain bounded function independent of σ and

$$\Delta(\partial_y) = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2}.$$

The function $\Phi(y,x,\sigma,\Lambda)$ we shall call Carleman's functions for the Helmholtz equation. For her holds following inequalities:

$$|\Phi(y,x,\sigma,i\Lambda)| \le C_1 r^{-1}\,exp\,\sigma(y_3^2 - x_3^2)\,,$$

$$\left|\frac{\partial\Phi(y,x,\sigma,i\Lambda)}{\partial y_k}\right| \le C_2 r^{-2}\,\sigma\,exp\,\sigma(y_3^2-x_3^2)\,,\ k=1,2,3, \tag{2.7}$$

$$\left|\frac{\partial^2\Phi(y,x,\sigma,i\Lambda)}{\partial y_k\,\partial y_j}\right| \le C_3 r^{-3}\,\sigma^2 exp\,\sigma(y_3^2-x_3^2)\,,\ k,j=1,2,3,$$

here $C_k = const\,,\ k=1,2,3.$

From Lemma 2.1 we obtain

Lemma2.2. The matrix $\Pi(y,x,\omega,\sigma)$ given by (2.3) and (2.5) is Carleman's matrix for problem (1.1), (1.3).

By using (2.3), (2.4) and inequalities (2.7) we obtain

$$\int_{\partial D\setminus S}\left(|\Pi(y,x,\omega,\sigma)|+\left|R\left(\partial_y,n(y)\right)\Pi(y,x,\omega,\sigma)\right|\right)ds_y\le C(D)\sigma^2 exp(-\sigma x_3^2)\,, \qquad (2.8)$$

where $C(D)$ is a bounded function inside of D.

Let us set

$$2U_\sigma(x)=\int_S\left(\Pi(y,x,\omega,\sigma)\left\{R\left(\partial_y,n(y)\right)U(y)\right\}-\right.$$
$$\left.-\left\{\tilde R\left(\partial_y,n(y)\right)\tilde\Pi(y,x,\omega,\sigma)\right\}^*U(y)\right)ds_y\,,\quad x\in D. \qquad (2.9)$$

The following theorem holds.

Theorem2.2. Let $U(x)$ be a regular solution of the system (1.1) in D such that

$$|U(y)|+\left|R\left(\partial_y,n(y)\right)U(y)\right|\le M\,,\ \ y\in\partial D\setminus S. \qquad (2.10)$$

Then for $\sigma\ge 1$ the following estimate is valid:

$$|U(x)-U_\sigma(x)|\le MC(x)\sigma^2 exp(-\sigma x_3^2)\,,$$

Where

$$C(x)\int_{\partial D}\frac{ds_y}{r^2}.$$

Since, by formulas (2.2) and (2.9) we have

$$|U(x)-U_\sigma(x)|\le\frac{1}{2}\left|\int_{\partial D\setminus S}\left(\Pi(y,x,\omega,\sigma)\left\{R\left(\partial_y,n(y)\right)U(y)\right\}-\right.\right.$$
$$\left.\left.-\left\{\tilde R\left(\partial_y,n(y)\right)\tilde\Pi(y,x,\omega,\sigma)\right\}^*U(y)\right)ds_y\right|\le$$
$$\le\frac{1}{2}\int_{\partial D\setminus S}\left(|\Pi(y,x,\omega,\sigma)|+\left|\left\{\tilde R\left(\partial_y,n(y)\right)\tilde\Pi(y,x,\omega,\sigma)\right\}^*\right|\right)\left(|U(y)|+\left|R\left(\partial_y,n(y)\right)U(y)\right|\right)ds_y.$$

Now on the basis of (2.8) and (2.10) we obtain the required estimate.

Corollary. Provided theorem we have the following equivalent formulas continue

$$U(x)=\lim_{\sigma\to\infty}U_\sigma(x)=\frac{1}{2}\lim_{\sigma\to\infty}\int_S\left(\Pi(y,x,\omega,\sigma)\left\{R\left(\partial_y,n(y)\right)U(y)\right\}-\right.$$
$$\left.-\left\{\tilde R\left(\partial_y,n(y)\right)\tilde\Pi(y,x,\omega,\sigma)\right\}^*U(y)\right)ds_y\,,\quad x\in D\,, \qquad (2.11)$$

$$U(x)=\frac{1}{2}\int_S\left(\Pi(y,x,\omega)\left\{R\left(\partial_y,n(y)\right)U(y)\right\}-\right.$$
$$\left.-\left\{\tilde R\left(\partial_y,n(y)\right)\tilde\Pi(y,x,\omega)\right\}^*U(y)\right)ds_y+\frac{1}{2}\int_0^\infty Q(x,\omega,\sigma)\,d\sigma\,,\quad x\in D.$$

Where

$$Q(x,\omega,\sigma)=\int_S \Big(P(y,x,\omega,\sigma)\left\{R\left(\partial_y,n(y)\right)U(y)\right\}-$$
$$-\left\{\tilde R\left(\partial_y,n(y)\right)\tilde P(y,x,\omega,\sigma)\right\}^* U(y)\Big)\,ds_y\,,\quad x\in D,$$
$$P(y,x,\omega,\sigma)=\frac{\partial}{\partial\sigma}\varPi(y,x,\omega,\sigma)=\left[\!\left[\frac{\partial}{\partial\sigma}\varPi_{kj}(y,x,\omega,\sigma)\right]\!\right]_{4\times4}.$$

$\varPi(y,x,\omega)$ matrix constructed according to the formula (2.3) and (2.4) at

$$\Phi(y,x,i\Lambda)=\frac{\exp(i\Lambda r)}{4\pi r}\;.$$

Equivalence formulas continuation (2.11) and (2.12) follows from the formula

$$\lim_{\sigma\to\infty}U_\sigma(x)=\int_0^\infty \frac{dU_\sigma(x)}{d\sigma}\,d\sigma+U_0(x)$$

Based on the continuation of the formula (2.11) and (2.12) we give solvability criterion the Cauchy problem (1.1), (1.3).

Theorem2.3. Let $S\in C^2$, $f\in C^1(S)$, $g\in C(S)$. then, for problem (1.3) to be

Solvable, it is necessary and sufficient that

$$\left|\int_0^\infty \partial_x^p Q(x,\omega,\sigma)\,d\sigma\right|<\infty\,,\quad |p|\le 2\,,$$

Where $p-$ a multi-index, uniformly on any compact $K\subset D$, $x\in K$.

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