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Time Evolution of the Survival Probability at Very Long Times

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Abstract: *The time behavior of a wave packet, transmitted through a potential barrier, is investigated for very long times assuming it to be initially located outside the barrier. We study the decay process of an unstable quantum system, especially in the case of the zero-energy resonance. We also derive the quantitative condition that this regime of the decay process takes place and discuss what kind of system is suitable to observe the decay. As follows from the time behavior of survival probability, there are two power laws of time that characterize these regimes both with and without the zero-energy resonance.*

Keywords: Time-dependent tunneling; survival probability; wave packet; multibarrier resonance.

1. Introduction

The decay of unstable quantum systems has attracted much attention in the early days of quantum mechanics. Quantum decays of unstable isolated systems appear in a wide variety of fields ranging from quantum physics [1-3], chemistry [4] to cosmology [5]. While the exponential decay law is everywhere occurred in nature, microscopic systems should be described by quantum mechanics and the exponential decay cannot be valid for all times; in particular, it undoubtedly fails when the time of evolution is large [1,6,7] or small [7,8]. Indeed, rather general considerations assure that the quantum decay law can be described by a three-step function: a Gaussian law at short times, an exponential law at intermediate times, and a power law at longer times [1-8]. It is correct to say that we have a very good understanding of the first few moments in the life of an unstable quantum state. This is not the case for the long-time tail when many attempts to produce evidence of post-exponential decay have been made with little success. Hence, further theoretical and experimental investigations of the decay law are required. Much attention has been lately paid to the asymptotic behavior of wave packets, moving in a free space [9-14], as a simple example of initial state decay. If the Gaussian wave packet is chosen as the initial state, the survival probability decreases asymptotically as t^{-1}. However, it is not necessarily valid for an arbitrary initial wave packet [9]. Actually, a decrease, being either slower or faster than this law, can occur for the wave packet which vanishes at zero momentum [10]. In addition, a study was made on the transmission of a Gaussian wave packet through a potential barrier [11-13]. It has been shown that the survival probability behaves itself as t^{-3} at very long times.

In this paper, we consider the properties of the survival probability in multibarrier resonance systems using an exact analytical approach that involves a double rectangular barrier. The survival probability $S(t)$ shows clearly the inverse power law of time t^{-n}, where n depends on the feature of the unstable system. We conclude that $n=3$ is just the case without the zero-energy resonance. We show that near the resonance the S(t) exhibits at long times a different behavior; instead of the well known $S(t)\sim t^{-3}$ power law we obtain $S(t)\sim t^{-1}$ in this case. Our conclusions are of general character and independent of the barrier form.

2. The Quantum Description of Decays

We first consider some relevant results from the stationary and non-stationary descriptions of quantum tunneling. In the stationary state description, a particle of mass m and momentum $\hbar k$ is incident on a real potential barrier $V(x)$. The fundamental system of solutions of the stationary Schrödinger equation is the two linearly independent wave functions $\Phi_L(x,k)$ and $\Phi_R(x,k)$, determined uniquely by the boundary conditions at $x \to \pm\infty$ [15]. We combine the two wave functions Φ_L and Φ_R into a vector with the components $\Phi_j(x,k)$ labeled by $j = 0,1$

$$\Phi_0(x,k) = \frac{1}{\sqrt{2}}\Phi_L(x,k), \quad \Phi_1(x,k) = \frac{1}{\sqrt{2}}\Phi_R(x,k) \tag{1}$$

Let the particle be described by the wave packet in a fully time-dependent treatment. Consider now an ensemble of a large number of identically prepared single-particle scattering events. In each, a particle with the same initial wave function, $\Psi(x,0)$, is incident from the left

$$\Psi(x,0) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{4}} \exp\left[-\left(\frac{x-x_i}{2\sigma}\right)^2 + ik_i x\right], \tag{2}$$

where $\sigma^2 =< \Delta x^2 >$. In other words, the wave packet, localized at $t = 0$ near x_i in front of the barrier, moves towards positive x-positions with the average momentum $\hbar k_i$.

Our method for finding a wave-packet solution is to expand the initial packet in a complete set of the scattering states

$$\Psi(x,0) = \int_{-\infty}^{\infty} [A_0(k)\Phi_0(x,k) + A_1(k)\Phi_1(x,k)]\frac{dk}{2\pi}, \tag{3}$$

where the weight amplitude $A_j(k)$ is defined as

$$A_j(k) = \int_{-\infty}^{\infty} \Psi(x,0)\Phi_j(x,k)^* dx, \quad j = 0,1 \tag{4}$$

Since the time-dependent Schrödinger equation is linear in the derivative with respect to time, a specification of the wave function at the initial time is sufficient to determine this function at any future time. The most general solution is obtained immediately as

$$\Psi(x,t) = \int_{-\infty}^{\infty} [A_0(k)\Phi_0(x,k) + A_1(k)\Phi_1(x,k)]\exp\left(-i\frac{\hbar k^2}{2m}t\right)\frac{dk}{2\pi} \tag{5}$$

Let the system under consideration exist at $t = 0$ in the $\Psi(x,0)$ state. The potential barrier affects the motion of the system as perturbation and results in the $\Psi(x,t)$ state for the future time. Our prime interest here is with the decay of the initial state. This problem amounts precisely to the determination of the survival probability that the system at time t will be still in the unperturbed state. In the case of a closed system, the desired survival amplitude is the overlap integral of perturbed and unperturbed wave packets [6,7]

$$A(t) = \int_{-\infty}^{\infty} \Psi(x,0)^* \Psi(x,t) dx \tag{6}$$

The square of its modulus is the survival probability $S(t)$. It is identical to the quantum fidelity between the initial state and the time-evolving state [2]. We now substitute Eqs. (2) and (5) into (6) to obtain an explicit expression for the survival amplitude

$$A(t) = \int_{-\infty}^{\infty} e^{-i\lambda(t)k^2} \varphi(k) dk, \tag{7}$$

where $\lambda(t) = \hbar t / 2m$ and $\varphi(k)$ is the normalized probability distribution over wave numbers

$$\varphi(k) = \frac{1}{2\pi}\left(|A_0(k)|^2 + |A_1(k)|^2\right) \tag{8}$$

Note that the survival probability is uniquely determined by the momentum distribution $\varphi(k)$. This is an even function of the momentum $\hbar k$.

3. Power Decay of the Survival Probability at Long Times

We next focus our attention on the following integral expression for $\exp(-i\lambda(t)k^2)$ with $\lambda(t) > 0$, namely:

$$e^{-i\lambda(t)k^2} = \frac{e^{-i\pi/4}}{\sqrt{4\pi\lambda(t)}} \int_{-\infty}^{\infty} \exp\left(i\frac{x^2}{4\lambda(t)} + ikx \right) dx \tag{9}$$

Formally, the integral can be represented as a series whose terms are expressed in terms of the derivatives of delta-function

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ikx} \sum_{n=0}^{\infty} \left(\frac{ix^2}{4\lambda(t)} \right)^n \frac{1}{n!} = \sum_{n=0}^{\infty} \left(\frac{-i}{4\lambda(t)} \right)^n \frac{\delta^{(2n)}(k)}{n!} \tag{10}$$

The action of the $\delta^{(2n)}(k)$ generalized function on the $\varphi(k)$ function yields the solution of our problem. Equations (7) and (9), in conjunction with Eq. (10), are asymptotically equivalent to

$$A(t) = \sqrt{4\pi} e^{-i\pi/4} \sum_{n=0}^{\infty} \frac{(-i)^n \varphi^{(2n)}(0)}{n!(4\lambda(t))^{n+1/2}} \tag{11}$$

As one might expect, there is a strong correlation between the behavior of the momentum distribution in the vicinity of zero momentum and the asymptotic temporal behavior of the survival amplitude. The presence of such correlation has been observed earlier [10].

We see that one of the remarkable properties of the survival probability is its power decay law at very long times. First, we consider the case $\varphi(0) = 0$ when the term with $n = 0$ is absent in this asymptotic series. This result is obtained due to the fact that the wave function $\Phi_j(x,k)$ is equal to zero before and after the barrier, and also it is equal to zero inside the barrier due to the continuity condition at the barrier boundaries. Physically, this means that incident plane waves with $k \approx 0$ completely reflected by the barrier. This is exactly the case without the zero energy resonance [15] (see below Section 4). Therefore, from Eq. (11) we obtain an asymptotic formula for the survival probability

$$S(t) = \frac{\pi}{16\lambda(t)^3} \left(\varphi^{(2)}(0) \right)^2 + \frac{\pi}{1024\lambda(t)^5} \left[\left(\varphi^{(4)}(0) \right)^2 - \frac{4}{3} \varphi^{(2)}(0) \varphi^{(6)}(0) \right] + \dots \tag{12}$$

The power decay was predicted for systems with a continuous energy spectrum bounded from below [1,6]. For the particles, tunneling through the barrier, this limitation becomes a needless condition because a particle can move both towards the barrier (with $k > 0$) and in the opposite direction (with $k < 0$). In addition, the two cases should be always considered for an asymmetric barrier where the particles move to the barrier both from the left and from the right which is taken into account by two terms in Eq. (8). Let us now pay attention to a particular power law t^{-3} in Eq. (12). This decay law has been earlier determined for several models [7,11], and derived analytically in a different way [12,13].

Let $V(x)$ be a finite-range potential satisfying $V(x) = 0$ for $|x| > b$. All the expansion coefficients of the wave function and the reflection amplitude can be expressed in terms of the transmission amplitude $T(k)$. At the $k \to 0$ limit, the following bifurcation is known to hold [15]: $T(0) = 0$ for the generic case, and $T(0) = -1$ for the exceptional case where the potential supports a zero-energy solution, $\Phi_j(x, k = 0) \neq 0$. In that case, the probability distribution defined by Eqs. (4) and (8) is not equal to zero at $k = 0$. Hence $S(t)$ has the asymptotic form

$$S(t) = \frac{\pi}{\lambda(t)} \varphi(0)^2 + \frac{\pi}{16\lambda(t)^3} \left[\left(\varphi^{(2)}(0) \right)^2 - \varphi(0)\varphi^{(4)}(0) \right] + \dots \tag{13}$$

The decay is determined by the power law t^{-1} in this case.

Thereupon the question arises of how $S(t)$ will look like for a free particle. The answer depends on the choice of the initial wave function [9]. Let the wave packet be set as (2). In obvious notation, the survival amplitude is of the form

$$A(t)=\int_{-\infty}^{\infty}\Psi(x,0)^{*}dx\int_{-\infty}^{\infty}\left(\frac{m}{2\pi i\hbar t}\right)^{\frac{1}{2}}\exp\left[-\frac{m(x-x')^2}{2i\hbar t}\right]\Psi(x',0)dx' \tag{14}$$

Then we find from Eq. (14)

$$S(t)=\left(1+\frac{t^2}{t_0^2}\right)^{-\frac{1}{2}}\exp\left[-\frac{4(k_i\sigma)^2}{1+t^2/t_0^2}\left(\frac{t}{t_0}\right)^2\right], \tag{15}$$

where $t_0=4m\sigma^2/\hbar$ is the characteristic time for the free motion. At very short times, the survival probability is proportional to the exponent with quadratic time dependence. At $t>>t_0$ the survival probability decays by the t^{-1} law [14]. This is an instructive example of the system for which the decay law t^{-3} is not applicable and the exponential regime is absent at all t.

4. TUNNELING THROUGH THE DOUBLE RECTANGULAR BARRIER

Here we treat the exactly solvable model of a double rectangular potential barrier to show the example of a decaying process with different power laws at very long times. An analytic representation for the barrier is of the form $V_0\Theta(|x|-a)\Theta(b-|x|)$. A well of width $2a$ is located between two equal barriers, each of width $b-a$ and height V_0. Exact results for this problem were obtained previously [16,17]. The transmission amplitude is given by (in our notation)

$$T(k)=-e^{-2ik(a+b)}\frac{1-|\rho(k)|^2}{1-\rho(k)^2e^{-4ika}}, \tag{16}$$

where

$$\rho(k)=\frac{k^2+\kappa^2+2ik\kappa ctg[\kappa(b-a)]}{\kappa^2-k^2}, \tag{17}$$

$$\kappa=\left(k^2-\kappa_V^2\right)^{\frac{1}{2}}\quad and \quad \kappa_V^2=\frac{2mV_0}{\hbar^2} \tag{18}$$

The perfect transmission is sure to arise when the transmission probability satisfies the condition, $|T(k)|^2=1$. This requirement based on (16) may be expressed by

$$tg(2ka)=\frac{2k\kappa}{k^2+\kappa^2}ctg[\kappa(b-a)] \tag{19}$$

For fixed values of a, b and V_0, we get a finite number of resonances under the barrier and, in principle, an infinite number of resonances in the over-barrier region. The expansion of $T(k)$ to the second order in k gives an expression of the form

$$T(k)=-\frac{\left[1-\coth^2(\kappa_V(b-a))\right]k^2}{D(k)}\left(1-2ik(a+b)-2k^2(a+b)^2\right), \tag{20}$$

where

$$D(k)=ik\kappa_V\left[\coth(\kappa_V(b-a))+\kappa_V a\right]+k^2\left[1+\coth^2(\kappa_V(b-a))\right]+2k^2\kappa_V^2a^2+4k^2\kappa_V a\coth(\kappa_V(b-a)) \tag{21}$$

Substituting $k=0$, we find that the above expression tends to a finite value not equal to zero, if the following condition is fulfilled

$$\coth(\kappa_V(b-a))+\kappa_V a=0 \tag{22}$$

This is the condition of the zero-energy resonance. The solution of this equation exists only for negative V_0 values (two potential wells). When (22) is satisfied, $T(0)=-1$ in this limiting case. For the double barrier under study, we find $T(0)=0$.

5. Conclusion

In this paper, we demonstrate that the survival probability of an unstable initial state can decrease by the two power laws at very long times, in accordance with its feature at the zero momentum. We consider the double rectangular barrier and show that the survival probability exhibits the asymptotic behavior t^{-3}, while another power law t^{-1} can also appear for two potential well.

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