

On the Eigen Value Problem in Rindler Space

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Abstract: In this article in a very general manner we have investigated the eigen value problem in Rindler space. We have developed the formalism in an exact form. It has been noticed that although the Hamiltonian is non-hermitian, because of the PT-symmetric nature, the eigen values are real, where P and T are the parity operator and the time reversal operator respectively. It has further been observed that the eigen energies are linearly quantized and the binding of the system increases with the increase in the strength of uniform gravitational field although its origin is purely classical.

1. INTRODUCTION

It is well known that the conventional Lorentz transformations are the space-time coordinate transformations between two inertial frame of references [1]. However, following the principle of equivalence, it is trivial to obtain the space-time transformations between a uniformly accelerated frame and an inertial frame and vice-versa in the same manner as it is done in special theory of relativity [2-6]. In the present scenario the flat space-time geometry is called the Rindler space. For the sake of illustration of principle of equivalence, one may state, that a reference frame undergoing an accelerated motion in absence of gravity is equivalent to a frame at rest in presence of gravity. Therefore in the present picture, the magnitude of the uniform acceleration is exactly equal to the strength of constant gravitational field. It may be assumed that the gravitational field is produced by a strong gravitating object. We further approximate that the gravitational field is constant within a small domain of special region. Since it is exactly equal to the uniform acceleration of the moving frame, this is also called the local acceleration of the frame.

The article has been organized in the following manner: In the next section We have reviewed the formalism to obtain the expression for particle Hamiltonian, when observed from a frame undergoing uniform accelerated motion or in other words in Rindler space. In section III we have given our formalism on the exact solution of the eigen value equation and finally in section IV, we have discussed the conclusion of this work.

2. SPECIAL RELATIVITY IN RINDLER SPACE

In this section, for the sake of completeness, following the references [7-9] we shall establish some of the useful formulas of special theory for a uniformly accelerated frame of reference. Before we go to the scenario of uniform acceleration of the moving frame, let us first assume that the frame S' has rectilinear motion with uniform velocity v along x-direction with respect to some inertial frame S . Further the coordinates of an event occurred at the point P (say) is indicated by (x, y, z, t) in S -frame and with (x', y', z', t') in the frame S' . The primed and the un-primed coordinates are related by the conventional form of Lorentz transformations and are given by

$$\begin{aligned} x' &= \gamma(x - vt), \quad y' = y, \quad z' = z \quad \text{and} \\ t' &= \gamma(t - vx) \quad \text{with } \gamma = (1 - v^2)^{-\frac{1}{2}} \end{aligned} \quad (1)$$

is the well known Lorentz factor. Throughout this article we have followed the natural system of units, i.e., speed of light in vacuum, $c = 1$ and later we put the Boltzmann constant $k_B = 1$ and the plank constant $\hbar = 1$. Next we consider a uniformly accelerated frame S' moving with uniform acceleration α also along x-direction with respect to S -frame. Then the Rindler coordinates are given by (see the references [7-9]),

$$t = \left(\frac{1}{\alpha} + x'\right) \sinh(\alpha t') \quad \text{and}$$

$$x = \left(\frac{1}{\alpha} + x'\right) \cosh(\alpha t') \tag{2}$$

Hence one can also express the inverse relations

$$t' = \frac{1}{2\alpha} \ln\left(\frac{x+t}{x-t}\right) \quad \text{and} \quad x' = (x^2 - t^2)^{\frac{1}{2}} - \frac{1}{\alpha} \tag{3}$$

The Rindler space-time coordinates as mentioned above are then just an accelerated frame transformation of the Minkowski metric of special relativity. The Rindler coordinate transformations change the Minkowski line element from

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \quad \text{to} \tag{4}$$

$$ds^2 = (1 + \alpha x')^2 dt'^2 - dx'^2 - dy'^2 - dz'^2 \tag{5}$$

Since the motion is assumed to be rectilinear along x-direction, $dy' = dy$ and $dz' = dz$. The form of the metric tensor can then be written as

$$g^{\mu\nu} = \text{diag}((1 + \alpha x)^2, -1, -1, -1) \tag{6}$$

Since we shall deal with the acceleration frame only, we have dropped the prime symbols. Now following the concept of kinematics of particle motion in special theory of relativity [1], the action integral may be written as (see also [10] and [11])

$$S = -\alpha_0 \int_a^b ds \equiv \int_a^b L dt \tag{7}$$

Then using eqns.(5) and (7) and putting $\alpha_0 = -m_0$ [1], where m_0 is the rest mass of the particle, the Lagrangian of the particle is given by [11]

$$L = -m_0[(1 + \alpha x)^2 - v^2] \tag{8}$$

Where v is the velocity of the particle. The momentum of the particle is then given by

$$p = m_0 v [(1 + \alpha x)^2 - v^2]^{-\frac{1}{2}} \tag{9}$$

Hence the Hamiltonian of the particle or the single particle energy is given by

$$H = L - pv = \epsilon(p) = m_0(1 + \alpha x) \left(1 + \frac{p^2}{m_0^2}\right)^{\frac{1}{2}} \tag{10}$$

This is the well known Rindler Hamiltonian. Here p and v are the particle momentum and velocity respectively along positive x-direction. In the inertial frame with $\alpha = 0$, we get back the results of special theory of relativity. In the case of classical mechanics, x, p and H are dynamical variables, whereas in the quantum mechanical picture, x, p and H are operators. In the quantum mechanical case x and p are also canonical conjugate of each other, i.e., $[x, p] = i\hbar$. Here it is quite obvious that the Hamiltonian operator represented by eqn.(10) is non-Hermitian. However from our subsequent analysis and discussion we will show that the eigen values or the eigen spectra are real in nature. This is found to be solely because of the PT symmetric nature of the Hamiltonian operator [12]. Under P and T operations we have the following relations from PT-symmetric quantum mechanics: $PxP^{-1} = -x, TxT^{-1} = x, PpP^{-1} = -p, TpT^{-1} = -p, PaP^{-1} = -\alpha, TaT^{-1} = \alpha$ and $=TiT^{-1} = -i$. The last relation is essential for the preservation of canonical quantization relation under PT operation, i.e., for the validity of $PT[x, p](PT)^{-1} = i\hbar$. Hence it is quite obvious to verify that for the Hamiltonian, $PTH(PT)^{-1} = H$, i.e., the Hamiltonian operator is PT invariant. We shall show in our subsequent discussion that since eigen functions Ψ are the functions of the product αx , which is PT symmetric, therefore $PT\Psi(u) = \Psi(u)$, where u is a function of the product of αx , i.e., $\Psi(u)$ is an eigen function of PT operator with eigen value +1.

3. EXACT SOLUTION OF EIGEN VALUE PROBLEM

In this section we shall develop a formalism to get exact solution for the relativistic form of quantum mechanical equation. In the present formalism we have also used the natural units, i.e., $\hbar = c = 1$. We begin with the classical Hamiltonian in the Rindler space, given by

$$H = (1 + \alpha x)(p^2 + m_0^2)^{\frac{1}{2}} \tag{11}$$

Where m_0 is the rest mass of the particle. Then the eigen value equation $H\Psi = E\Psi$ may be written as

$$(1 + \alpha x)(-d_x^2 + m_0^2)^{\frac{1}{2}}\Psi = E\Psi \tag{12}$$

Where $d_x = \frac{d}{dx}$ and we assume that the motion is one dimensional and along positive x-direction. Changing the variable from x to X , given by $X = 1 + \alpha x$, the above equation reduces to

$$X(-d_X^2 + m^{*2})^{\frac{1}{2}}\Psi = E^*\Psi \tag{13}$$

Where we have re-defined $m^* \rightarrow m_0 = \frac{m_0}{\alpha}$ and the energy eigen value $E^* \rightarrow E = \frac{E}{\alpha}$. For the new variable X , the limit is from 1 to ∞ . Then we can rewrite the above differential equation as

$$X(-d_X^2 + m_0^2)^{\frac{1}{2}}\Psi = E\Psi \tag{14}$$

To get an analytical solution, we follow the technique presented in [18-20]. Now using the properties of Dirac delta function, we can write [18-20] the left hand side of the above equation in the form

$$X(-d_X^2 + m_0^2)^{\frac{1}{2}}\Psi(X) = \int_{-\infty}^{+\infty} X(-d_X^2 + m_0^2)^{\frac{1}{2}}\delta(q - X)\Psi(q)dq \tag{15}$$

Since $\delta(x - a)f(x) = \delta(x - a)f(a)$, we have

$$X(-d_X^2 + m_0^2)^{\frac{1}{2}}\Psi(X) = \int_{-\infty}^{+\infty} q(-d_X^2 + m_0^2)^{\frac{1}{2}}\delta(q - X)\Psi(q)dq \tag{16}$$

Using the integral representation of δ -function, given by

$$\delta(q - X) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp \exp[-i(q - X)p] \tag{17}$$

Where the particle momentum p has been treated here as some kind of integration variable. Then we have

$$X(-d_X^2 + m_0^2)^{\frac{1}{2}}\Psi(X) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q(p^2 + m_0^2)^{\frac{1}{2}} \exp[-i(q - X)p]\Psi(q)dpdq \tag{18}$$

Hence without the loss of generality, we can re-write the right hand side in the following form

$$X(-d_X^2 + m_0^2)^{\frac{1}{2}}\Psi(X) = \frac{1}{2\pi} (-d_X^2 + m_0^2) \int_{-\infty}^{+\infty} q\Psi(q)dq \int_{-\infty}^{+\infty} dp \frac{\exp[-i(q-X)p]}{(p^2 + m_0^2)^{\frac{1}{2}}} \tag{19}$$

With some simple algebraic manipulation, the above equation can be expressed in the following form [17]

$$\begin{aligned} X(-d_X^2 + m_0^2)^{\frac{1}{2}}\Psi(X) &= \frac{1}{\pi} (-d_X^2 + m_0^2) \int_{-\infty}^{+\infty} q\Psi(q)dq \int_0^{\infty} dp \frac{\cos[(q-X)p]}{(p^2 + m_0^2)^{\frac{1}{2}}} \\ &= \frac{1}{\pi} (-d_X^2 + m_0^2) \int_{-\infty}^{+\infty} dq q\Psi(q)K_0[m_0|q - X|] \end{aligned} \tag{20}$$

Where $K_0(x)$ is the modified Bessel function of second kind of order zero. On decomposing the q integral into two parts, we have

$$X(-d_X^2 + m_0^2)^{\frac{1}{2}}\Psi(X) = \frac{1}{\pi}(-d_X^2 + m_0^2) * \left[\int_{-\infty}^X dq q\Psi(q)K_0[m_0(X-q)] + \int_X^{+\infty} dq q\Psi(q)K_0[m_0(q-X)] \right] \quad (21)$$

Then substituting $X - q = q_1$ in the first integral and $q - X = q_2$ in the second integral and redefining $q_1 = q$ in the first integral, and $q_2 = q$ in the second integral, we have

$$X(-d_X^2 + m_0^2)^{\frac{1}{2}}\Psi(X) = \frac{1}{\pi}(-d_X^2 + m_0^2) * \int_0^\infty dq K_0(m_0 q) [(X+q)\Psi(X+q) + (X-q)\Psi(X-q)] \quad (22)$$

We seek the serious solution for the wave function in the form

$$\Psi(X) = \sum_{k=1}^{n+1} \gamma_{k,n+1} X^k \exp(-\beta X) \quad (23)$$

Where $\gamma_{k,n+1}$ and β are unknown constants, to be obtained from the recursion relations. Then the n_{th} term is given by

$$X(-d_X^2 + m_0^2)^{\frac{1}{2}} X^n \exp(-\beta X) = \frac{1}{\pi}(-d_X^2 + m_0^2) \exp(-\beta X) * \int_0^\infty dq K_0(m_0 q) [(X+q)^{n+1} \exp(-\beta q) + \exp(\beta q)(X-q)^{n+1}] \quad (24)$$

Now expanding $(X+q)^{n+1}$ and $(X-q)^{n+1}$ in Binomial series and then using the standard relation we have [17]

$$\int_0^\infty x^{\mu-1} \exp(-\alpha x) K_\nu(\beta_1 x) dx = \frac{(\pi)^{\frac{1}{2}} (2\beta_1)^\nu \Gamma(\mu+\nu) \Gamma(\mu-\nu)}{(\alpha+\beta_1)^{\mu+\nu} \Gamma(\mu+\frac{1}{2})} F\left(\mu+\nu, \nu+\frac{1}{2}; \mu+\frac{1}{2}; \frac{\alpha-\beta_1}{\alpha+\beta_1}\right) \quad (25)$$

Where $F(a, b; c; d)$ is the Hypergeometric function and in our case with $\nu = 0, x = q, \mu - 1 = k + 1, \alpha = \beta$ and $\beta_1 = m_0$, the integral reduces to

$$I = \frac{(\pi)^{\frac{1}{2}} [\Gamma(k+2)]^2}{\Gamma(k+\frac{5}{2})(m_0+\beta)^{k+2}} F\left(k+2, \frac{1}{2}; k+\frac{5}{2}; -\frac{m_0-\beta}{m_0+\beta}\right) \quad (26)$$

Then after a little algebra, we have

$$X(-d_X^2 + m_0^2)^{\frac{1}{2}} X^n \exp(-\beta X) = \frac{1}{(\pi)^{\frac{1}{2}}} (-d_X^2 + m_0^2) \exp(-\beta X) \sum_{k=0}^{n+1} \binom{n+1}{k} G_k(m_0, \beta) X^{n+1-k} \quad (27)$$

Where

$$G_k(m_0, \beta) = \frac{[\Gamma(k+2)]^2}{\Gamma(k+\frac{5}{2})} \left[\frac{1}{(m_0+\beta)^{k+2}} F\left(k+2, \frac{1}{2}; k+\frac{5}{2}; -\frac{m_0-\beta}{m_0+\beta}\right) \right] + \frac{[\Gamma(k+2)]^2}{\Gamma(k+\frac{5}{2})} (-1)^k \left[\frac{1}{(m_0-\beta)^{k+2}} F\left(k+2, \frac{1}{2}; k+\frac{5}{2}; -\frac{m_0+\beta}{m_0-\beta}\right) \right] \quad (28)$$

Now it is a matter of simple algebra to show that

$$X(-d_X^2 + m_0^2) X^{n+1-k} \exp(-\beta X) = (m_0^2 - \beta^2) \exp(-\beta X) X^{n+1-k} + 2(n+1-k)\beta \exp(-\beta X) X^{n-k} - (n+1-k)(n-k) \exp(-\beta X) X^{n-k-1} \quad (29)$$

Then

$$X(-d_X^2 + m_0^2) X^n \exp(-\beta X) = \frac{1}{(\pi)^{\frac{1}{2}}} (-d_X^2 + m_0^2) \exp(-\beta X) \sum_{k=0}^{n+1} \binom{n+1}{k} [(m_0^2 - \beta^2) G_k(m_0, \beta) + 2\beta k G_{k-1}(m_0, \beta) - k(k-1) G_{k-2}(m_0, \beta) X^{n+1-k}] \quad (30)$$

Hence from the quantum mechanical equation

$$X(-d_X^2 + m_0^2)^{\frac{1}{2}}\Psi(X) = E\Psi(X) \tag{31}$$

We have from the series solution (polynomial form) of $\Psi(X)$,

$$\begin{aligned} X(-d_X^2 + m_0^2)^{\frac{1}{2}}\sum_{k=1}^{n+1}\gamma_{k,n+1}X^k \exp(-\beta X) &= \sum_{k=1}^{n+1}\gamma_{k,n+1} \sum_{p=0}^k F_{p,k}(m_0, \beta) X^{k+1-p} \\ &= E_n \sum_{k=1}^{n+1}\gamma_{k,n+1}X^k \exp(-\beta X) \end{aligned} \tag{32}$$

The expression for $F_{p,k}(m_0, \beta)$ and $\gamma_{k,n+1}$ can be defined in terms of the parameter β and the rest mass m_0 [18]. Form the above equation, equating the coefficient of x^2 , we have

$$E_n = \frac{\gamma_{1,n+1}}{\gamma_{2,n+1}} F_{0,n+1}(m_0, \beta) \tag{33}$$

The energy corresponding to the n_{th} level of the spectrum. The quantity $\gamma_{l,n}$ can be obtained from [18]. Further equating the coefficient of X^l from both the sides the putting $p = 1$, we have

$$\gamma_{l,n+1}(\beta, m_0)F_{1,l}(m_0, \beta) = E_l(\beta, m_0)\gamma_{l,n+1}(m_0, \beta) \quad \text{or} \tag{34}$$

$$E_l(m_0, \beta) = F_{1,l}(m_0, \beta) \tag{35}$$

Which gives the energy spectrum in terms of the unknown parameter β , quantum number l and the strength of gravitational field α . Where

$$F_{1,l}(m_0, \beta) = \frac{-l\beta\alpha}{(2\pi m_0)^{\frac{1}{2}}(m_0^2 - \beta^2)^{\frac{3}{4}}} P_{\frac{-l}{2}}^{\frac{-l}{2}}\left(\frac{\beta}{m_0}\right) \tag{36}$$

Where $l = 1, 2, 3, \dots$, positive integers. Therefore to obtain the energy spectrum, we have to evaluate the associated Legendre function. It is to be noted further that the energy spectrum is real and linearly quantized ($\propto l$). Now because of the factor $\exp(-\beta X)$ the wave functions are bounded and also the energy eigen values are negative in nature. From the above expression it is quite obvious that $\beta < m_0$. This is also a necessary condition for the argument z of $P_v^u(z)$ to be real. It has been noticed that the magnitude of the energy eigen value $|E_l(m_0, \beta)|$ increases with β and the rise is very sharp as $\beta \rightarrow m_0$. However, it is actually negative throughout and the negativity increases with β and also with α . Further the eigen functions are $\propto \exp(-\beta X)$, therefore with the increase of β the wave function converges to zero very quickly, whereas the eigen states become more bound because of high negative value of energy for large β . Or in other words, when $\frac{\beta}{m_0}$ is very close to unity, the nature of wave functions and energy eigen values indicate that the binding of states are strong enough. Therefore with the increase of negative value of the energy makes the state more bound and simultaneously the spread of wave function in space is reduced. The later is also in agreement with more stronger binding. At this point we would like to emphasis that with the increase in the strength of gravitational field produced by a black, i.e., as we approach more and more close to the event horizon, the absolute value of the energy eigen value increases more and more makes the system more bound. In our study since α is the gravitational field strength of the black hole, we may conclude that the increase in classical background gravitational field, which is assumed to be uniform locally, makes the particle, which is quantum mechanical in nature, more strongly bound. The binding is with the black hole in presence of strong classical background field.

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