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The Natures of Microscopic Particles Depicted by Nonlinear Schrödinger Equation in Quantum Systems

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Abstract: *When the microscopic particles was depicted by linear Schrödinger equation we find that the particles have only a wave feature, thus, a series of diffuculties and intense disputations occur in quantum mechanics. These problems excite us to have to consider the nonlinear interactions among the particles or between the particle and background field, which is really existed, but ignored completely in quantum mechanics. Thus use here the nonlinear Schrödinger equation containing the nonlinear interaction to describe the microscopic particles. In this case the natures and features of microscopic particles are studied completely and in detail. From these investigations we obtain a lot of new results. The microscopic particles described by the nonlinear Schrödinger equation are localized due to the nonlinear interaction, have truly a wave-particle duality which are also very stable, satisfy both the classical dynamics equation and Lagrangian and Hamilton equations, obey the collision rules of classical particles and possess the invariance and conservation laws of mass, energy and momentum and angular momentum for the macroscopic particles. At the same time, the microscopic particles possess also a wave feature and can generate the reflection and the transmission phenomenon at interfaces, which but are different from both KdV solitary wave and linear wave. Farthermore, the position and momentum of the particles satisfy a minimum uncertainty relation, which represents a wave-corpuscle feature of the particles. These natures and features are completely different from those in the quantum mechanics. Thus we can affirm the theory established on the basis of nonlinear Schrödinger equation is a new quantum theory, the linear quantum mechanics is only its a especial case. The new quantum theory can not only solve difficulties and problems disputed for about a century by plenty of scientists in quantum mechanics but also promote the development of physics and enhance the knowledge and recognition levels to the essences of microscopic matter.*

Keywords: *microscopic particle, nonlinear interaction, quantum mechanics, nonlinear systems, nonlinear Schrödinger equation, nonlinear theory, wave-particle duality, motion rule*

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1. INTRODUCTION, PHYSICAL BACKGROUND

It is well known that several great scientists, such as Bohr, Born, Schrödinger and Heisenberg, etc. established quantum mechanics in the early 1900s[1-6] , which is the foundation and pillar of modern science and provides an unique way of describing the properties and rules of motion of microscopic particles (MIP) in microscopic systems. In quantum mechanics the state of microscopic particles is described by the Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+V\left(\vec{r},t\right)\psi \tag{1}$$

where is the kinetic energy operator, V(,t) is the externally applied potential operator, m is the mass of particles, is a wave function describing the states of particles, is the coordinate or position of the particle, and t is the time. This theory states that once the externally applied potential field and initial states of the microscopic particles are given, the states of the particles at any time later and any position can be determined by the Schrödinger equation (1) in the case of nonrelativistic motion. In this theory the Hamiltonian operator of the system corresponding Eq.(1) is

$$\hat{H}(t)=\hbar^2\nabla^2/2m+V(\vec{r},t) \tag{2}$$

where $T=\hbar^2\nabla^2/2m$ is the kinetic energy operator, V the external potential energy operator.

The quantum mechanics has achieved a great success in descriptions of motions of microscopic particles, such as, the electron, phonon, exciton, polaron, atom, molecule, atomic nucleus and elementary particles, and in predictions of properties of matter based on the motions of these quasi-particles. For example, energy spectra of atoms (such as hydrogen atom, helium atom), molecules (such as hydrogen molecules) and compounds, electrical, optical and magnetic properties of atoms and condensed matters can be calculated based on linear quantum mechanics and the calculated results are in basic agreement with experimental measurements. Thus considering that the quantum mechanics is thought of as the foundation of modern science, then the establishment of the theory of quantum mechanics has revolutionized not only physics, but also many other science branches such as chemistry, astronomy, biology, etc., and at the same time created many new branches of science, for instance, quantum statistics, quantum field theory, quantum electronics, quantum chemistry, quantum optics and quantum biology, etc. Therefore, we can say the quantum mechanics has achieved a great progress in modern science. One of the great successes of linear quantum mechanics is the explanation of the fine energy spectra of hydrogen atom, helium atom and hydrogen molecule. The energy spectra predicted by the quantum mechanics are in agreement with experimental data. Furthermore, new experiments have demonstrated that the results of the Lamb shift and superfine structure of hydrogen atom and the anomalous magnetic moment of the electron predicted by the theory of quantum electrodynamics are in agreement with experimental data. It is therefore believed that the quantum electrodynamics is one of the successful theories in modern physics.

Studying the above postulates in detail, we can find [7-13] that the quantum mechanics has the following characteristics.

(1) Linearity. The wave function of the particles, $\psi(\vec{r},t)$, satisfies the linear Schrödinger equation (1) and linear superposition principle, that is, if two states, $|\psi_1\rangle$ and $|\psi_2\rangle$ are both eigenfunctions of a given linear operator, then their linear combination holds:

$$|\psi\rangle = C_1|\psi_1\rangle + C_2|\psi_2\rangle$$

where C_1 and C_2 are constants relating to the state of these particle. The operators are some linear operators in the Hilbert space. This means that the quantum mechanics is a linear theory, thus it is quite reasonable to refer to the theory as the linear quantum mechanics.

(2) The independence of Hamiltonian operator on the wave function. From Eq. (2) we see clearly that the Hamiltonian operator of the systems is independent on the wave function of state of the particles, in which the interaction potential contained relates also not to the state of the particles. Thus the potential can change only the states of the particles, such as the amplitude, but not its natures. Therefore, the natures of the particles can only be determined by the kinetic energy

term, $T = \hbar^2\nabla^2/2m$ in Eqs. (1) and (2).

(3) The wave feature. The Schrödinger equation (1) is in essence a wave equation and has only wave solutions, which do not include any corpuscle feature. In fact, let the wave function be $\psi = f\exp[-iEt/\hbar]$ and substitute it into Eq. (1), we can obtain

$$\partial^2 f\big/\partial x^2 + k_0^2 n^2 f = 0$$

where $n^2 = (E-U)/(E-C) = k^2/k_0^2$, C is a constant, $k_0^2 = 2m(E-C)/\hbar^2$. This equation is nothing but that of a light wave propagating in a homogeneous medium. Thus, the linear Schrödinger equation (1) is unique one able to describe the wave feature of the microscopic particle. In other words, when a particle moves continuously in the space-time, it follows the law of linear variation and disperses over the space-time in the form of a wave of microscopic particles. Therefore, the linear Schrödinger equation (1) is a wave equation in essence, thus the microscopic particles are only a wave. This is a basic or essential nature of the microscopic particles in quantum mechanics.

This nature of the particles can be also verified by using the solutions of Eq. (1)[7-18]. In fact, at $V(\vec{r},t)$ =0, its solution is a plane wave:

$$\psi(\vec{r},t) = A'\exp[i(\vec{k}\cdot\vec{r} - \omega t)] \qquad (3)$$

where k, ω, A' and are the wave vector, frequency, and amplitude of a wave, respectively. This solution denotes the state of a freely moving microscopic particle with an eigenenergy:

$$E = \frac{p^2}{2m} = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2), (-\infty < p_x, p_y, p_y < \infty) \tag{4}$$

This is a continuous spectrum. It states that the probability of the particle to appear at any point in the space is same, thus a microscopic particle propagates freely in a wave and distributes in total space, this means that the microscopic particle cannot be localized and has nothing about corpuscle feature.

If a free particle can be confined in a small finite space, such as, a rectangular box of dimension a, b and c, the solution of Eq. (1) is standing waves as follows:

$$\psi(x,y,z,t) = A\sin\left(\frac{n_1\pi x}{a}\right)\sin\left(\frac{n_2\pi y}{b}\right)\sin\left(\frac{n_3\pi z}{c}\right)e^{-iEt/\hbar}$$

where n_1, n_2 and n_3 are three integers. In this case, the particle is still not localized, it appears also at each point in the box with a determinant probability. In this case the eigenenergy of the particle in this case is quantized as follows:

$$E = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} + \frac{n_3^2}{c^2}\right)$$

where n_1, n_2 and n_3 are some integers. The corresponding momentum is also quantized. This means that the wave feature of microscopic particle has not been changed because of the variation of itself boundary condition

If the potential field is further varied, for example, the microscopic particle is subject to a conservative time-independent field, $V(\vec{r},t) = V(\vec{r}) \neq 0$, then the microscopic particle satisfies the time-independent linear Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi' + V(\vec{r})\psi' = E\psi'$$

Where

$$\psi = \psi'(\vec{r})e^{-iEt/\hbar}$$

When $V = \vec{F}\cdot\vec{r}$, here $\vec{F}$ is a constant field force, such as, a one dimensional uniform electric field E′, then $V(x) = -eE'x$, thus its solution is

$$\psi' = A\sqrt{\bar{\xi}}H_{1/2}^{(1)}\left(\frac{2}{3}\bar{\xi}^{3/2}\right), \left(\bar{\xi} = \frac{x}{l} + \bar{\lambda}\right)$$

where $H^{(1)}(x)$ is the first kind of Hankel function, A is a normalized constant, l is the characteristic length, and $\bar{\lambda}$ is a dimensionless quantity. The solution remains a dispersed wave. When $\xi \to \infty$, it approaches

$$\psi'(\xi) = A'\xi^{-1/4}e^{-2\xi^{3/2}/3}$$

to be a damped wave.

If $V(x) = ax^2$, the eigenenergy and eigenwave function are

$$\psi'(x) = N_n e^{-a^2x^2/2}H_n(\alpha x)$$

and

$$E_n = (n + \frac{1}{2})\hbar\omega, \ (n=0,1,2,\ldots),$$

respectively, here $H_n(\alpha x)$ is the Hermite polynomial. The solution obviously has a decaying feature. If the potential fields are successively varied, we find that the wave nature of the solutions in Eq. (1) does not change no matter what the forms of interaction potential. This shows clearly that the wave nature of the particles is intrinsic in quantum mechanics.

(4) Simplicity. We can easily solve arbitrary complicated quantum problems or systems, only if their potential functions are obtained. Therefore, to solve quantum mechanical problems becomes almost to find the representations of the external potentials by means of various approximate methods.

(5) Quantization. The particles, which the matter is composed of , are quantized in the microscopic systems. Concretely, the eigenvalues of physical quantities of the particles are quantized. For instance, the eigenenergy of the particles at $V(\vec{r},t)=0$ in Eq. (1) is quantized, when $V(x)=ax^2$, its eigenenergy, $E_n=(n+\frac{1}{2})\hbar\omega$, is also quantized as mentioned above, and so on. In practice, the momentum, moment of momentum, and spin of the microscopic particles are all quantized in quantum mechanics. These quantized effects refer to as microscopic quantum effects because they occur on the microscopic scale.

Because of the above nature of quantum mechanics, some novel results, such as the uncertainty relationship between the position and momentum and the mechanical quantities are denoted by some average values in an any state, occur also. Meanwhile, the wave nature of the particles obtained from this theory is not only incompatible with de Broglie relation, $E=h\upsilon=\hbar\omega$ and $\vec{p}=\hbar\vec{k}$, of wave-corpuscle duality for microscopic particles and Davisson and Germer's experimental result of electron diffraction on double seam in 1927[12-13], but also contradictory to the traditional concept of particles. Thus a lot of difficulties and problems occur in the quantum mechanics, among them the central problem is how we represent and delineate the corpuscle feature of the microscopic particles. Aimed at this issue, Born introduce a statistic explanation for the wave function, and use $\left|\psi(\vec{r},t)\right|^2$ to represent the probability of the particles occurring the position $\vec{r}$ at time t in the space-time. However, the microscopic particles have a wave feature and can disperse over total system, thus the probability $\left|\psi(\vec{r},t)\right|^2$ has a certain value at every point, for example, the probability of the particle denoted by Eq. (3) is same at all points. This means that the particle can occur at every point at same time in the space. In this case, a fraction of particle must appear in the systems, which is a very strange phenomenon and is quite difficult to understand. However, in physical experiments, the particles are always captured as a whole one not a fractional one by a detector placed at an exact position. Therefore, the concept of probability representing the corpuscle behavior of the particles cannot be accepted.

Due to the linearity of the theory and the dispersive effect of the microscopic particles, then the use of quantum mechanics is impossible to describe the corpuscle feature of microscopic particles. Thus some scientists suggest that using a wave packet, for example, a Gaussian wave packet, represents the corpuscle behavior of a particle. The wave packet is given by

$$\psi(x,t=0)=A_0\exp[-\beta_0^2x^2/2] \tag{5}$$

at t = 0, where A_0 is a constant. Although this wave packet is localized at t = 0 because $|\psi|^2=|A_0|^2\exp[-\beta_0^2x^2]\to 0$ at $x\to\infty$, the wave packet is also inappropriate to denote the corpuscle feature of the particles because it disperses and attenuates always with time during the course of propagation, that is

$$\psi(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(k)\exp[i(kx-\hbar k^2t/2m]dk$$

$$=\frac{1}{\beta_0\sqrt{\beta_0^{-2}+i\hbar t/m}}\exp[-x^2(\beta_0^{-2}+i\hbar t/m)/2]$$

where

$$\Psi(k,t=0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x,t=0)e^{ikx}dx$$

In this case,

$$|\psi(x,t)|^2 = \frac{1}{\sqrt{1+(\beta_0^2\hbar t/m)^2}} e^{-x^2/\beta_t^2} \text{ , } \beta_t = \frac{1}{\beta_0}\sqrt{1+i\beta_0^2\hbar t/m}$$

This indicates clearly that the wave packet is instable and dispersed as time goes by, and its position is also uncertain with the time, which is not the feature of corpuscle. The corresponding uncertainty relation is

$$\Delta x \Delta p = \frac{\hbar}{2}\sqrt{1+\beta_0^4\hbar^2 t^2/m^2}$$

where

$$\Delta x = \frac{1}{2\beta_0}\sqrt{1+\beta_0^4\hbar^2 t^2/m^2}, \Delta p = \frac{\hbar\beta_0}{\sqrt{2}}$$

Hence, the wave packet cannot be applied to describe the corpuscle property of a microscopic particle. How to solve this problem has always been a challenge in the quantum mechanics. This is just an example of intrinsic difficulties of the quantum mechanics.

On the other hand, we know from Eqs.(1)-(2) that the quantum mechanics requires to incorporate all interactions among particles or between particles and background field, such as the lattices in solids and nuclei in atoms and molecules, including nonlinear and complicated interactions, into the external potential by means of various approximate methods, such as, the free electron and average field approximations, Born-Oppenheimer approximation, Hartree-Fock approximation, Thomas- Fermi approximation, and so on. This is obviously incorrect. The method replacing these real interactions by an average field amounts to freeze or blot out real motions and interactions of the microscopic particles and background fields, which was often used in the quantum mechanics to study the properties of the particles in the systems of many-particles and many-bodies. This indicates that the quantum mechanics is an approximate theory and can be only used in some sample atoms, such as hydrogen atom, cannot use to solve the real properties of the microscopic particles in complicated atoms, condensed matters, polymers and biological systems. In contrast, since the electrons denoting by wave function $\psi(\vec{r},t)$ in atoms is a wave, then it does not have a determinant position in quantum mechanics, but the vector $\vec{r}$ is use to denote the position of the electron with charge e and mass m in the wave function and the Coulemb potential, $V(r)=-Ze^2/\vec{r}$. Thus it is difficult to understand correctly these contradictory representations in quantum mechanics.

These difficulties and problems of the quantum mechanics mentioned above inevitably evoked the contentions and further doubts about the theory among physicists. Actually, taking a closer look at the history of physics, we could find that not so many fundamental assumptions were required for a physical theory but the linear quantum mechanics. Obviously, these assumptions of linear quantum mechanics caused its incompleteness and limited its applicability. However, the disputations continued and expanded mainly between the group in Copenhagen School headed by Bohr representing the view of the main stream and other physicists, including Einstein, de Broglie, Schrödinger, Lorentz, etc[7-16]

However, why does quantum mechanics have these questions? This is worth studying deeply and in detail. As is known, dynamic equation (1) describes the motion of a particle and Hamiltonian operator of the system, Eq.(2), consist only of kinetic and potential operator of particles; the potential is only determined by an externally applied field, and not related to the state or wave function of the particle, thus the potential can only change the states of MIP, and cannot change its nature and essence. Therefore, the natures and features of MIP are only determined by the kinetic term. Thus there is no force or energy to obstruct and suppress the dispersing effect of kinetic energy in the system, then the MIP disperses and propagates in total space, and cannot be localized at all. This is the main reason why MIP has only wave feature in quantum mechanics. Meanwhile, the Hamiltonian in Eq.(2) does not represent practical essences and features of MIP. In real physics, the energy operator of the systems and number operator of particles are always associated with the states of particles, i.e., they are related to the wave function of MIP. On the other hand, Eq.(1) or (2) can describe only the states and feature of a single particle, and cannot describe the states of many particles. However, a system composed of one particle does not exist in nature. The simplest system in nature is the hydrogen atom, but it consists of two particles. In such a case, when we study the states of particles in realistic systems composed of many particles and many bodies using quantum mechanics, we have to use a simplified and uniform average-potential unassociated with the states of particles to replace the complicated and nonlinear interaction among these particles [10-11]. This means that the real motions of the microscopic particles and background field as well as the interactions, including the nonlinear interactions, between them are completely frozen in such a case. Thus, these complicated effects and nonlinear interactions determining essences and natures of particles[17-30] are ignored completely. This is obviously not reasonable and correct. In this case the nature of microscopic particles can be only determined by the kinetic energy term in Eq.(1). Therefore, the microscopic particles described by quantum mechanics possess only a wave feature, not corpuscle feature. This is just the essence of quantum mechanics. Then we can only say that quantum mechanics is an approximate and linear theory and cannot represent completely the properties of motion of MIPs. Just so, we here refer to it as linear quantum mechanics (LQM)[17-30]. Meanwhile, a lot of hypotheses or theorems of particles in quantum mechanics also do not agree with conventional understanding, and have excited a long-time debate between scientists. Up to now, there is no unified conclusion. Therefore, it is necessary to improve and develop LQM.

2. THE NONLINEAR INTERACTION OF MICROSCOPIC PARTICLES AND CORRESPONDING NONLINEAR SCHRÖDINGER EQUATION

From the above studies we know that an essential shortcoming or defect of LQM is to ignore the real motion of microscopic particles and background field as well as the nonlinear interactions among these particles or between the particles and background field, in which these real motions of particles and the influences of background field on the studied particles are replaced by an average field, thus these nonlinear interactions are completely rubbed away or ignored. Thus the nonlinear interactions occurs not in the dynamical equation (1). As a matter of fact, the nonlinear interactions always exist in any realistic physics systems including the hydrogen atom, if only the real motions of the particles and background as well as their interactions are completely considered[17-32]. Since the nonlinear interactions are completely ignored in linear quantum mechanics, thus the microscopic particles described Eqs.(1)-(2) have not corpuscle feature, only wave feature. This means that the nonlinear interactions play a key rule in determining their corpuscle feature and localizations. In order to this point we now study the effects of dispersion and nonlinear interaction[28-32].

We now study carefully the motion of water wave in sea. When a water wave approaches the beach, its shape variants gradually from a sinusoidal cross section to triangular, and eventually a crest which moves faster than the rest. This is a result of the nonlinear nature of wave, i.e., nonlinear interaction deform the wave. We show the changes of a wave with increasing the nonlinear interaction in Fig.1, where $\xi = x - vt$. As the water wave approaches the beach the wave will be broken up due to the fact that the nonlinear interaction is enhanced[33-34]]. Since the

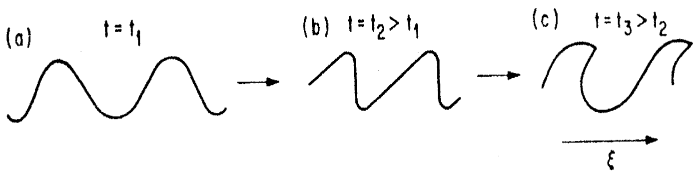

Fig1. *The changes of wave with increasing nonlinear interactions*

speed of wave propagation depends on the height of the wave in such a case, so, this is a nonlinear phenomenon. If the phase velocity of the wave, v_c, depends weakly on the height of the wave, h, then $v_c = \frac{\omega}{k} = v_{co} + \Theta_1 h$, where $\Theta_1 = \frac{\partial v_c}{\partial h}\Big|_{h=h_0}$, h_0 is the average height of the wave surface, v_{co} is the linear part of the phase velocity of the wave, Θ_1 is a coefficient denoting the nonlinear effect. Therefore, the nonlinear interaction results in changes in both form and velocity of waves. This is the same for the dispersion effect, but their mechanism and rules are different. When the dispersive effect is weak, the velocity of a wave is denoted by $v_c = \frac{\omega}{k} = v_{co}^{'} + \Theta_2 k^2$, where v_{co} is a dispersionless phase velocity, $\Theta_2 = \frac{\partial^2 v_c}{\partial k^2}\Big|_{k=k_0}$ is the coefficient of the dispersion feature of the wave. Generally speaking, the lowest order dispersion occurring in the phase velocity is proportional to k^2, and the term proportional to k gives rise to the dissipation effect. If the two effects act simultaneously on a wave, then it is necessary to change the nature of the wave.

To further explore the effects of nonlinear interaction on the behaviors of microscopic particles, we consider a simple motion as follows[33-34]

$$\phi_e + \phi\phi_x = 0 \tag{6}$$

where $\phi\phi_x$ is a nonlinear interaction. There is no dispersive term in this equation. It is easy to verify that $\phi = \Phi^{'}(x - \phi t)$ satisfies Eq.(4). This solution indicates that as time elapses, the front side of wave gets steeper and steeper, until it becomes triple-valued function of x due to the nonlinear interaction, which does not occur for a general wave equation. This is a deformation effect of wave resulting from the nonlinear interaction. If let $\phi = \Phi^{'} = \cos \pi x$ at $t = 0$, then at $x = 0.5$ and $t = \pi^{-1}, \phi = 0$ and $\phi_x = \infty$. The time $t_B = \pi^{-1}$ at which the wave becomes very steep is called destroyed period of the wave. However, the collapsing phenomenon can be suppressed by adding a dispersion term ϕ_{xxx} as in the KdV equation[33-34]. Then, the system has a stable soliton, $\sec h^2(X)$, in such a case. Therefore, a stable soliton, or a localization of particle can occur only if the nonlinear interaction and dispersive effect exist simultaneously in the system, and can be balanced and canceled each other. Otherwise, the particle cannot be localized, and a stable soliton cannot be formed.

However, if ϕ_{xxx} is replaced by ϕ_{xx}, then Eq. (6) becomes

$$\phi_t + \phi\phi_x = v\phi_{xx}, (v > 0) \tag{7}$$

This is the Burgur's equation. In such a case, the term $v\phi_{xx}$ cannot suppress the collapse of the wave, arising from the nonlinear interaction $\phi\phi_{xx}$. Therefore, the wave is damped. In fact, utilizing the Cole-Hopf transformation $\phi = -2\gamma \frac{d}{dx}(\log \psi^{'})$, equation (5) becomes $\frac{\partial \psi^{'}}{\partial t} = v \frac{\partial^2 \psi^{'}}{\partial x^2}$. This is a linear equation of heat conduction or diffusion equation, which has a damping solution. Therefore, the Burgur's equation (7) is not a equation with soliton solution[33-34].

This example shows that the deformational effect of nonlinearity on the wave can suppress its dispersive effect, thus a soliton solution of dynamic equations can then occur in such a case [33-34]. Or speaking, if the nonlinear interaction and dispersive effect occur simultaneously in a dynamical equation, then its solution is a soliton having wave and corpuscle features, in which the nonlinear term sharpens the peak of wave, while its dispersion term has the tendency to leave it off, thus the solution of dynamical equation is a soliton.

These results tell us that if the nonlinear interactions among the microscopic particles or between the particles and background field are considered, then the dynamic equation of microscopic particles is no longer the Schrödinger equation (1), but the following nonlinear Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\phi = -\frac{\hbar^2}{2m}\nabla^2\phi + V(x,t)\phi - b|\phi|^2\phi \tag{8}$$

where we use $\phi(\vec{r},t)$ to denotes the state of the microscopic particle, b is a nonlinear interaction coefficient, $b|\phi|^2\phi$ denotes just the nonlinear interaction among the particles or between the particle and background field. We study now the natures and features of microscopic particles described by the nonlinear Schrödinger equations (8).

Obviously, the Lagrange density function corresponding to Eq.(8) is given as follows[23-27]:

$$\mathcal{L}' = \frac{i\hbar}{2}\left(\phi^*\phi_t - \phi\phi_t^*\right) - \frac{\hbar^2}{2m}\left(\nabla\phi \cdot \nabla\phi^*\right) - V(x)\phi^*\phi + (b/2)\left(\phi^*\phi\right)^2 \tag{9}$$

where L' = L is the Lagrange density function. The momentum density of the particle system is defined as P= ∂ L/$\partial\phi$. Thus, the Hamiltonian density of the systems is as follows

$$H' = \frac{i\hbar}{2}\left(\phi^*\partial_t\phi - \phi\partial_t\phi^*\right) - \mathcal{L} = \frac{\hbar^2}{2m}\left(\nabla\phi\cdot\nabla\phi^*\right) + V(x)\phi\phi^* - (b/2)\left(\phi\phi^*\right)^2 \qquad (10)$$

where H' = H is the Hamiltonian density. Equations.(9)-(10) show clearly that the Lagrange density function and Hamiltonian density of the systems are all related to the wave function of state of the particles and involve a nonlinear interaction, (b/2) $(\phi\phi^*)^2$. From the above fundamental principles, we see clearly that the new theory breaks through the fundamental hypotheses of the LQM in two aspects, the linearity of dynamic equations and independence of the Hamiltonian operator with the wave function of the particles. In the new theory, the dynamic equations are a nonlinear partial differential equation, in which nonlinear interactions, $b|\phi|^2\phi$, related to state wave function ϕ are involved; the Hamiltonian and Lagrangian operators in Eqs.(9)-(10) corresponding to the equation are all related to the state wave function ϕ. Hence, so far as this point is concerned, the new theory[23-27] is really a break-through or a new development in quantum mechanics. In the theory the natures of microscopic particles are simultaneously determined by the kinetic and nonlinear interaction terms. Hence, we refer to the systems as nonlinear quantum mechanics systems. We expect[33-34] that the nonlinear interaction could suppress and balance the dispersive effect of kinetic energy of the particles in dynamics equations and make the particles be localized as soliton with wave-corpuscle feature. However, the nonlinear Schrödinger equation is evolved from linear Schrödinger equation in linear quantum mechanics. Therefore, the new theory is a development of linear quantum mechanics.

3. THE WAVE-CORPUSCLE FEATURES OF SOLUTIONS OF NONLINEAR SCHRÖDINGER EQUATION

3.1. The Wave-Corpuscle Duality of Solutions of Simple Nonlinear Schrödinger Equation

As it is known, the microscopic particles have only the wave feature, but not corpuscle property in the quantum mechanics. Thus, it is very interesting what are the properties of the microscopic particles in the nonlinear quantum mechanics? We now study firstly the properties of the microscopic particles described by nonlinear Schrödinger equation in Eq.(8). In the one-dimensional case, the equation (8) at V(x,t)=0 becomes as

$$i\phi_{t'} + \phi_{x'x'} + b|\phi|^2\phi = 0 \qquad (11)$$

where $x' = x/\sqrt{\hbar^2/2m}$, $t' = t/\hbar$. We now assume the solution of Eq.(11) to be of the form

$$\phi(x',t') = \varphi(x',t')e^{i\theta(x',t')} \qquad (12a)$$

or

$$\phi(x',t') = \varphi(\xi')e^{i\Omega t' + iQx'} \qquad (12b)$$

where $\theta = \theta(\varsigma = x' - v_c t'), \varphi = \varphi(\xi' = x' - v_e t'),$. Inserting Eq. (12a) into Eq.(11) we can obtain

$$\varphi_{\xi'\xi'}(\xi') + i(2Q - v_e)\varphi_{\xi'} - (\Omega + Q^2)\varphi(\xi') - b\varphi^2\varphi = 0, (b > 0) \qquad (13)$$

If the imaginary coefficient of $\varphi_{\xi'}$ vanishes, then $Q = v_e/2$ .Then from $A = Q^2 + \Omega$ we get that $\Omega = A - v_e^2/4$. Thus from Eq.(13) we obtain

$$\varphi_{\xi'\xi'} + b\varphi^3 - A\varphi = 0 \qquad (14)$$

The equation (14) can be integrated, which results in

$$(\varphi_{\xi'})^2 = D + A\varphi^2 - b\varphi^4/2 \qquad (15)$$

where D is an integral constant. The solution $\varphi(\xi')$ of Eq.(15) is obtained by inverting an elliptic integral:

$$\int_{\varphi_0}^{\varphi} \frac{d\varphi}{\sqrt{D + A\varphi^2 - b\varphi^4/2}} = \pm\xi' \qquad (16)$$

Let $P(\varphi')=(\alpha_1-\varphi'^2)(\varphi'^2-\alpha_2)=-\varphi'^4+A(2/b)^{1/2}\varphi'^2+D$, where $\varphi'=(b/2)^{1/4}\varphi$, from Eq.(16) we can get $[K(k)-F(\varphi',k)]=\pm\xi'$, where K(k) and $F(\varphi',k)$ are the first associated elliptic integral and incomplete elliptic integral, respectively, and $k=[(\alpha_1-\alpha_2)/\alpha_1]^{-1/2}$,

$\alpha_{1,2}=A/(2b)^{1/2}\pm[D+(A^2/2b)]^{1/2}$. Using these and $\alpha=\varphi'^2_{1,2}$, we have

$$\varphi'[(b/2)^{1/4}\xi']=\varphi_1'\{1-[(1-\varphi_1'^2/\varphi_2'^2)sn^2(\xi'(b/2)^{1/4},k)]\}^{1/2} \tag{17}$$

When $D\to 0,\varphi_1'\to\varphi_0',k\to 1,\varphi'\to\varphi_0'\sec h[\varphi_0'\xi'(b/2)^{1/4}]$, where $\varphi_0'=(2A^2/b)^{1/4}$, the soliton solution of Eq.(11) can be finally represented by

$$\phi_s(x',t')=\sqrt{\frac{2A}{b}}\sec h[\sqrt{A}(x'-v_et')]\exp[i(v_ex'/2-v_e^2t'/4+A(1/b)^{1/2}t')]$$

(18)

Pang[18,23-27]represented eventually the solution of nonlinear Schrödinger equation in Eq. (11) in the coordinate of (x,t) by

$$\phi(x,t)=A_0\sec h\left\{\frac{A_0\sqrt{bm}}{\hbar}\left[(x-x_0)-vt\right]\right\}e^{i[mv(x-x_0)-Et]/\hbar} \tag{19}$$

where $A_0=\sqrt{(mv^2/2-E)/2b}$, v is the velocity of motion of the particle, $E=\hbar\omega$. This solution is completely different from Eq. (3), and consists of a envelop and carrier waves, the former is $\varphi(x,t)=A_0\sec h\left\{A_0\sqrt{bm}\left[(x-x_0)-vt\right]/\hbar\right\}$ and a bell-type non-topological soliton with an amplitude A_0, the latter is the $\exp\{i[mv(x-x_0)-Et]/\hbar\}$. This solution is shown in Fig.2a. Therefore, the particles described by nonlinear Schrödinger equation (11) are solitons. The envelop φ(x, t) is a slow varying function and is a mass centre of the particles; the position of the mass centre is just at x_0, A_0 is its amplitude, and its width is given by $W'=2\pi\hbar/A_0\sqrt{2m}$.Thus, the size of the particle is $A_0W'=2\pi\hbar/\sqrt{2m}$ and a constant. This shows that the particle has exactly a determinant size and is localized at x_0. Its form resemble a wave packet, but differ in essence from both the wave solution in Eq. (1) and the wave packet mentioned above in linear quantum mechanics due to invariance of form and size in its propagation process. According to the soliton theory[33-34], the bell-type soliton in Eq. (19) can move freely over macroscopic distances in a uniform velocity *v* in space-time retaining its form, energy, momentum and other quasi-particle properties. However, the wave packet in linear quantum mechanics is not so and will be decaying and dispersing with increasing time. Just so, the

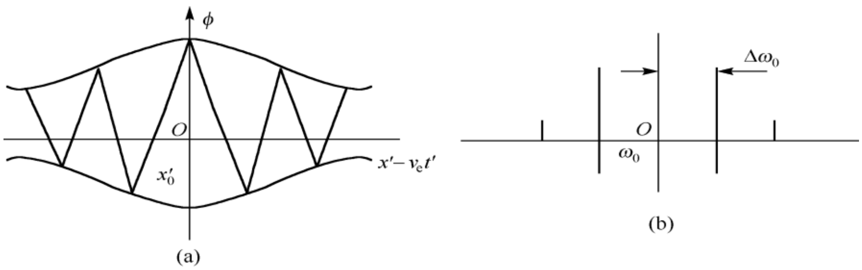

Fig2. *The solution of Eq. (11) and its features*

vector $\vec{r}$ or x in the representation in Eq.(19) has definitively a physical significance, and denotes exactly the positions of the particles at time *t*. Thus, the wave- function $\phi(\vec{r},t)$ or φ(x,t) can represent exactly the states of the particle at the position $\vec{r}$ or *x* at time *t*. These features are consistent with the concept of particles. Thus the microscopic particles depicted by Eq. (11) display outright a corpuscle feature.

Using the inverse scattering method Zakharov and Shabat[35-36] obtained also the solution of Eq. (11) , which was represented as

$$\varphi(x',t') = 2\left(\frac{2}{b}\right)^{1/2}\eta \operatorname{sech}\left[2\eta\left(x'-x_0'\right)+8\eta\xi t'\right]\exp\left[-4i\left(\xi^2-\eta^2\right)t'-i2\xi x'-i\theta\right] \tag{20}$$

in the coordinate of (x',t'), where η is related to the amplitude of the microscopic particle, ξ relates to the velocity of the particle, $\theta = \arg\gamma, \lambda = \xi + i\eta, x_0' = (2\eta)^{-1}\log(|\gamma|/2\eta), \gamma$ is a constant.. We now re-write it as following form[23-27]:

$$\phi(x',t') = 2\sqrt{\frac{2}{b}}k \operatorname{sech}\left\{2k\left[\left(x'-x_0'\right)-v_e t'\right]\right\}e^{iv_e[(x'-x_0')-v_c t']/2} \tag{21}$$

where v_e is the group velocity of the electron, v_c is the phase speed of the carrier wave in the coordinate of (x',t'). For a certain system, v_e and v_c are determinant and do not change with time. We can obtain $2^{3/2}k/b^{1/2}= A_0$, $A_0 = \sqrt{\dfrac{v_e^2-2v_c v_e}{2b}}$. According to the soliton theory, the soliton shown in Eq. (21) has determinant mass, momentum and energy, which can be represented by[23-27]

$$N_s = \int_{-\infty}^{\infty}|\phi|^2 dx' = 2\sqrt{2}A_0\ ,\ p = -i\int_{-\infty}^{\infty}\left(\phi^*\phi_{x'} - \phi\phi_{x'}^*\right)dx' = 2\sqrt{2}A_0 v_e = N_s v_e = const\ ,$$

$$E = \int_{-\infty}^{\infty}\left[\left|\phi_{x'}\right|^2 - \frac{1}{2}|\phi|^4\right]dx' = E_0 + \frac{1}{2}M_{sol}v_e^2 \tag{22}$$

where $M_{sol} = N_s = 2\sqrt{2}A_0$ is just effective mass of the particles, which is a constant. Thus we can confirm that the energy, mass and momentum of the particle cannot be dispersed in its motion, which embodies concretely the corpuscle features of the microscopic particles. This is completely consistent with the concept of classical particles. This means that the nonlinear interaction, $b|\phi|^2\phi$, related to the wave function of the particles, balances and suppresses really the dispersion effect of the kinetic term in Eq. (11) to make the particles become eventually localized. Thus the position of the particles, $\vec{r}$ or x, has a determinately physical significance.

However, the envelope of the solution in Eqs. (19)- (21) is a solitary wave. It has a certain wavevector and frequency as shown in Fig.2(b), and can propagate in space-time, which is accompanied with the carrier wave. Its feature of propagation depends on the concrete nature of the particles. Figure 2(b) shows the width of the frequency spectrum of the envelope φ(x,t) which has a localized distribution around the carrier frequency ω_0. This shows that the particle has also a wave feature[23-27]. Thus we believe that the microscopic particles described by nonlinear quantum mechanics have simultaneously a wave-corpuscle duality. Equations (19) - (21) and Figure 1.a are just the most beautiful and perfect representation of this property, which consists also of de Broglie relation, $E = h\upsilon = \hbar\omega$ and $\vec{p} = \hbar\vec{k}$, wave-corpuscle duality and Davisson and Germer's experimental result of electron diffraction on double seam in 1927 as well as the traditional concept of particles in physics [11-13].

3.2. The Wave-Corpuscle Duality of the Nonlinear Schrödinger Equation Solutions with Different Potentials

We can verify that the nature of wave-corpuscle duality of microscopic particles is not changed when varying the externally applied potentials. As a matter of fact, if $V(x') = \alpha x'$+c in Eq. (8), where α and c are some constants, in this case Pang [37-39] replaced Eq. (13) by

$$\varphi_{x'x'} - \varphi\theta_{t'} - \varphi\theta_{x'}^2 - b\varphi^2\varphi = \alpha x' + c. \tag{23}$$

Now let

$$\varphi(x',t') = \varphi(\xi), \xi = x' - u(t'), u(t') = -\alpha(t')^2 + vt' + d \tag{24}$$

where $u(t')$ describes the accelerated motion of $\varphi(x'.t')$. The boundary condition at $\xi \to \infty$ requires $\varphi(\xi)$ to approach zero rapidly, then equation (14) can be written as

$$-\dot{u}\frac{\partial\varphi}{\partial\xi}+2\frac{\partial\varphi}{\partial\xi}\frac{\partial\theta}{\partial\xi}+\varphi\frac{\partial^2\theta}{\partial\xi^2}=0 \tag{25}$$

where $\dot{u}=\frac{du}{dt'}$. If $2\partial\theta/\partial\xi-\dot{u}\neq 0$, equation (25) may be denoted as

$$\varphi^2=\frac{g(t')}{(\partial\theta/\partial\xi-\dot{u}/2)} \quad \text{or} \quad \frac{\partial\theta}{\partial x'}=\frac{g(t')}{\varphi^2}+\frac{\dot{u}}{2} \tag{26}$$

Integration of Eq. (26) yields

$$\theta(x',t')=g(t')\int_0^{x'}\frac{dx''}{\varphi^2}+\frac{\dot{u}}{2}x'+h(t') \tag{27}$$

where $h(t')$ is an undetermined constant of integration. From Eq. (27) Pang[37-39] can get

$$\frac{\partial\theta}{\partial t'}=\dot{g}(t')\int_0^{x'}\frac{dx''}{\varphi^2}-\frac{g\dot{u}}{\varphi^2}+\frac{g\dot{u}}{\varphi^2}\Big|_{x'=0}+\frac{\ddot{u}}{2}x'+\dot{h}(t') \tag{28}$$

Substituting Eqs.(27)- (28) into Eq.(22), we have:

$$\frac{\partial^2\varphi}{\partial(x')^2}=\left[(ax'+c)+\frac{\ddot{u}}{2}x'+\dot{h}(t')+\frac{\dot{u}^2}{4}+\dot{g}\int_0^{x'}\frac{dx''}{\varphi^2}+\frac{g\dot{u}}{\varphi^2}\Big|_{x'=0}\right]\varphi-b\varphi^3+\frac{g^2}{\varphi^3} \tag{29}$$

Since $\frac{\partial^2\varphi}{\partial(x')^2}=\frac{d^2\varphi}{d\xi^2}$, which is a function of ξ only. In order for the right-hand side of Eq.(29) is also a function of ξ only, it is necessary that $g(t')=g_0=\text{constant}$, and

$$(ax'+c)+\frac{\ddot{u}}{2}x'+\dot{h}(t')+\frac{\dot{u}^2}{4}+\frac{g\dot{u}}{\varphi^2}\Big|_{x'=0}=\overline{V}(\xi) \tag{30}$$

Next, we assume that $V_0(\xi)=\overline{V}(\xi)-\beta$, where β is real and arbitrary, then

$$ax'+c=V_0(\xi)-\frac{\ddot{u}}{2}x'+\left[\beta-\frac{g\dot{u}}{\varphi^2}\Big|_{x'=0}-\dot{h}(t')-\frac{\dot{u}^2}{4}\right] \tag{31}$$

Clearly, in the discussed case $V_0(\xi)=0$, and the function in the brackets in Eq.(31) is a function of t′. Substituting Eq. (31) into Eq.(29), we can get:

$$\frac{\partial^2\varphi}{\partial\xi^2}=\beta\varphi-b\varphi^3+\frac{g_0^2}{\varphi^3} \tag{32}$$

where β is a real parameter and defined by

$$V_0(\xi)=\overline{V}(\xi)-\beta \tag{33}$$

with

$$(\alpha x'+c)+\frac{\ddot{u}}{2}x'+\dot{h}(t')+\frac{\dot{u}^2}{4}+\frac{g\dot{u}}{\varphi^2}\Big|_{x'=0}=\overline{V}(\xi) \tag{34}$$

Clearly, in the discussed case, $V_0(\xi)=0$. Obviously, $\varphi=\varphi(\xi)$ is the solution of Eq. (32) when β and g are constant. For large $|\xi|$, we may assume that $|\varphi|\leq\beta/|\xi|^{1+\Delta}$, when Δ is a small constant. To ensure that $d^2\varphi/d\xi^2$, and φ approach zero when $|\xi| \to \infty$, only the solution corresponding to $g_0=0$ in Eq. (32) is kept stable. Therefore we choose $g_0=0$ and obtain the following from Eq. (26)

$$\frac{\partial\theta}{\partial x'} = \frac{\dot{u}}{2} \tag{35}$$

thus, we obtain from Eq. (34)

$$\alpha x' + c = -\frac{\ddot{u}}{2}x' + \beta - \dot{h}(t') - \frac{\dot{u}^2}{4}, \quad h(t') = (\beta - v^2/4 - c)t' - \alpha^2 t'^3/3 + v\alpha t'^2/2]\} \tag{36}$$

Substituting Eq. (36) into Eqs.(28) and (34), we obtain

$$\theta = (-\alpha t' + v/2)x' + (\beta - v^2/4 - c)t' - \alpha^2 t'^3/3 + v\alpha t'^2/2]\} \tag{37}$$

Finally, substituting the above into Eq. (33), we can get

$$\frac{\partial^2\varphi}{\partial\xi^2} - \beta\varphi + b\varphi^3 = 0 \quad . \tag{38}$$

When $\beta > 0$, Pang gives the solution of Eq. (38), which is of the form

$$\varphi = \sqrt{\frac{2\beta}{b}}\,\mathrm{sech}(\sqrt{\beta}\xi) \tag{39}$$

Pang[37-39] finally obtained the complete solution in this condition, which is represented as

$$\begin{aligned}\phi(x',t') &= \sqrt{\frac{2\beta}{b}}\,\mathrm{sech}\left\{\sqrt{\beta}\left[\left(x' - x_0'\right) + (\alpha t'^2 - vt' - d)\right]\right\}\times \\ &\exp\{i[(-\alpha t' + v/2)\left(x' - x_0'\right) + (\beta - v^2/4 - c)t' - \alpha^2 t'^3/3 + v\alpha t'^2/2]\}\end{aligned} \tag{40}$$

This is a soliton solution. If V(x')=c, the solution can be represented as

$$\phi(x',t') = \sqrt{\frac{2\beta}{b}}k\,\mathrm{sech}\left\{\sqrt{\beta}\left[\left(x' - x_0'\right) - v(t' - t_0')\right]\right\}\exp\{i[v\left(x' - x_0'\right)/2 - (\beta - v^2/4 - c)t']\} \tag{41}$$

At $V(x') = 2\alpha x'$ and $b = 2$, we can also get a corresponding soliton solution from the above process. However, Chen and Liu [40-41] adopted the following transformation

$$\phi(x',t') = \phi'(x',\tilde{t}')\exp[-2i\alpha x'\tilde{t}' + 8i\alpha^2\tilde{t}'^3/3],\ x' = x' - 2\alpha\tilde{t}'^2,\ t' = \tilde{t}' \tag{42}$$

to make Eq. (5) at $A(\phi) = 0$ become

$$i\phi'_{\tilde{t}'} + \phi'_{\tilde{x}'\tilde{x}'} + 2|\phi'|^2\phi' = 0\,. \tag{43}$$

Thus Chen and Liu[40-41] represented the solution of Eq.(8) at $V(x') = ax'$, $b = 2$ by

$$\begin{aligned}\phi(x',t') &= 2\eta\,\mathrm{sech}\left\{2\eta\left[\left(x' - x_0'\right) + (2\alpha t'^2 - 4\xi't')\right]\right\}\exp\{-i[2(\xi' - \alpha t')\left(x' - x_0'\right) + \\ &+4(\xi'^2 - \eta^2)t' + 4\alpha^2 t'^3/3 - 4\xi'\alpha t'^2] + \theta_0\}\end{aligned} \tag{44}$$

At the same time, utilizing the above method Pang[25-27,37] found also the soliton solution of Eq. (8) and $V(x) = kx^2 + A(t)x + B(t)$, which could be represented as

$$\phi = \varphi(x - u(t))e^{i\theta(x,t)} \tag{45}$$

where $\varphi(x',t') = \sqrt{\frac{2\beta}{b}}\,\mathrm{sech}\left\{\sqrt{\beta}\left[\left(x' - x_0'\right) - u(t')\right]\right\}$, $u(t') = 2\cos(2\sqrt{k}t + \gamma) + u_0(t)$ (46)

$$\theta(x',t') = [-2\alpha\sin(2\sqrt{k}t'+\gamma)+u_0(t')/2)](x'-x_0') + \int_0^{t'}\{[-\alpha^2(2\cos(2\sqrt{k}t''+\gamma)+u_0(t'')/2)]^2 - B(t'')+[-2\alpha\sin(2\sqrt{k}t''+\gamma)+u_0(t'')/2]\}dt''+\bar{L}t'+\theta_0$$

where $\bar{L}$ is a constant related to A(t'). When $A(t)=B(t)=0$, the solution is still Eq. (45), but

$$u(t') = 2\cos(2\sqrt{k}t')+u_0(t')$$

$$\theta(x',t') = [-2\sqrt{k}\sin(2\sqrt{k}t')+u_0(t')/2)](x'-x_0') + \int_0^{t'}\{[-k(2\cos(2\sqrt{k}t'')+u_0(t'')/2)]^2+[-2\sqrt{k}\sin(2\sqrt{k}t'')+u_0(t'')/2]\}dt''+\theta_0$$

For the case of $V_0(x')=\alpha^2 x'^2$ and b=2, where α is constant, Chen and Liu[40-41] assume $u(t')=(2\xi/\alpha)\sin(2\alpha t')$, thus they represent the soliton solution in this condition by

$$\phi(x',t') = 2\eta\,\mathrm{sech}\{2\eta[(x'-x_0')-(2\xi'/\alpha)\sin(2\alpha t')]\}\exp\{i[2\xi'(x'-x_0')\cos 2\alpha(t'-t_0') - +4\eta^2(t'-t_0')-(\xi'^2/\alpha)\sin[4\alpha(t'-t_0')+\theta_0]\} \quad (47)$$

where $2\eta=\sqrt{\beta}$ is the amplitude of microscopic particles, $4\xi'$ is related to its group velocity in Eqs.(44) and (47), and ξ' is the same as ξ in Eq. (20). From the above results we see clearly that these solutions of nonlinear Schrödinger equation (8) under influences of different potentials, V(x)=c, $V(x')=\alpha x'$, $V(x')=\alpha x'+c$, $V(x)=kx^2+A(t)x+B(t)$ and $V(x')=\alpha^2x'^2$, still consist of envelop and carrier waves, which are analogous to Eq. (19), some bell-type solitons with a certain amplitude A_0, group velocity v_e and phase speed v_c, and have a mass center and determinant amplitude, width, size, mass, momentum and energy. If inserting these solutions, Eqs.(40), (41), (44), (46) and (47) into Eq. (21) we can find out the effective masses, moments and energies of these microscopic particles, respectively, which all have determinant values. Therefore we can determine that the microscopic particles described by these dynamic equations still possess a wave-corpuscle duality as shown in Fig.1, although they are acted by different external potentials. These potentials change only amplitude, size, frequency, phase and group and phase velocities of the particles, in which velocity and frequencies of some particles are further related to time and oscillatory. These results indicate that in Eq. (8) the kinetic energy term decides the wave feature of the particles, the nonlinear interaction determines its corpuscle feature, their combination results in its wave-corpuscle duality, but the external potentials influence only wave form, phase and velocity of particles, but cannot affect the wave-corpuscle duality. These results verify directly and clearly the necessity and correctness for describing the properties of microscopic particles using the nonlinear Schrödinger equation (8).

3.3. The Localization of Microscopic Particles and its Stability

3.3.1. The Localization of Microscopic Particles

However, how can microscopic particles described by the nonlinear quantum mechanics be localized? In order to answer this question, Pang[42-43] returned to discussing the property of nonlinear Schrödinger equation (8). We now represent it as the following form

$$i\hbar\frac{\partial\phi}{\partial t} = \hat{H}(\phi)\phi$$

where $\hat{H}(\phi) = -\frac{\hbar^2}{2m}\nabla^2 - b|\phi|^2 + V(\vec{r},t)$ is the Hamiltonian operator of the system. If we assume that the time-independent wave function of the particles is represented by

$$\phi(\vec{r},t) = \varphi'(\vec{r})e^{-iEt/\hbar} \quad (48)$$

and inserting it into the above equation Pang[15,42-43] obtained the following quasi-eigenenergy equation

$$\hat{H}(\varphi')\varphi' = E\varphi' \quad \text{or} \quad E\varphi'(\vec{r}) = -\frac{\hbar^2}{2m}\nabla^2\varphi'(\vec{r}) + V(\vec{r})\varphi'(\vec{r}) - b\left|\varphi'(\vec{r})\right|^2 \varphi'(\vec{r}) \tag{49}$$

If $V(\vec{r})$ and b are independent on $\vec{r}$, then equation (49) in a one-dimensional case may be written as

$$\frac{\hbar^2}{2m}\frac{\partial^2\varphi'}{\partial x^2} = -\frac{d}{d\varphi'}V_{eff}(\varphi') \tag{50}$$

where $V_{eff}(\varphi')$ is the effective potential of the system and can represent by[20-21]

$$V_{eff}(\varphi') = \frac{1}{4}b|\varphi'|^4 - \frac{1}{2}(V-E)|\varphi'|^2. \tag{51}$$

When $V > E$ and $V < E$, the relationship between $V_{eff}(\varphi')$ and φ' is shown in Fig.3. From this figure we see that there are two minimum values in the potential, which correspond to two ground states of the microscopic particle, *i.e*.

$$\varphi_0' = \pm\sqrt{(V-E)/b}. \tag{52}$$

Therefore the effective potential is a double-well potential and its energy is $-(V-E)^2/4b \le 0$. Thus the microscopic particle can be localized due to negative binding energy. This localization is achieved through repeated reflection of the microscopic particle in the double-well potential field. The two ground states limit the energy diffusion, thus the energy of the particle is gathered, a soliton is formed, and the particle is eventually localized. Obviously, this is due to the nonlinear interaction because the two ground states of the particles can occur in this case, only if the nonlinear interaction -b $(\varphi'\varphi'^*)^2 \neq 0$ or $b \neq 0$. In other words, once the nonlinear interaction eliminates, i.e, *b*=0, then the particles will disperse and have a common ground state, which is $\varphi' = 0$. This means that the nonlinear interaction plays a determinant and key role in the localization of the particles; its negatively localized energy is just provided by the attractive nonlinear interaction,-b$(\varphi'\varphi'^*)^2$, in Eq.(8). This demonstrates also that the wave feature of microscopic particles in quantum mechanics is due to absence of nonlinear interaction[42-43].

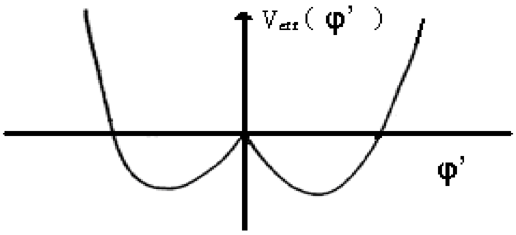

Fig3. *The effective potential of nonlinear Schrödinger equation*

However, when $V > 0$, $E > 0$ and $V < E$, or $|V| > E$, $E > 0$ and $V < 0$, for $b > 0$, the microscopic particles are not localized by those mechanisms because the systems cannot provide the double-well potential mentioned above in these conditions.

On the other hand, if the nonlinear interaction is repelling ($i.e, b < 0$), equation (5) with $A(\phi) = 0$ becomes

$$i\hbar\frac{\partial}{\partial t}\phi + \frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} - |b||\phi|^2\phi = V(x,t)\phi. \tag{53}$$

Although its effective potential is not a double-well potential mentioned above and the bell-type soliton in Eq. (19) cannot occurs as well, a dark or hole soliton can be formed in this case. Zakhorov and Shabat[35-36] obtained the dark soliton solution of Eq. (53) at $V(x) = 0$, which was experimentally observed in optical fiber and discussed in detail in Bose-Einstein condensation model by Barger et al[44].

On the other hand, if $V(x,t)=V(x)$ or is a constant, equation (53) has also a kink soliton solution. Inserting Eq. (48) into Eq. (53) we can get

$$\frac{\hbar^2}{2m}\frac{\partial^2\varphi'}{\partial x^2}-|b|\varphi'^3+\left[E-V(x)\right]\varphi'=0. \tag{54}$$

If V is independent of x and $0<V<E$, the kink solution can be represented by

$$\varphi'=\frac{\sqrt{2(E-V)}}{|b|}\tanh\left[\sqrt{\frac{2(E-V)}{\hbar^2}}(x-x_0)\right]. \tag{55}$$

If inserting Eq. (48) into Eq. (53) we can obtain another quasi-eigenequation:

$$E\varphi'=(H_0+|b||\varphi'|^2)\varphi'. \tag{56}$$

where $\hat{H}_0=-\frac{\hbar^2}{2m}\nabla^2+V(\vec{r})$. We can predict that the eigenenergy of the systems will lift, thus this model describes the nonlinear localization of another kind of particles, such as holes, in nonlinear semiconductors.

3.3.2. The Stability of Wave-corpuscle Duality of Localized Particles

Stability is an important property of the particles. What is the so-called stability of particles? In general, the stability points toward the fact that the particles cannot be dispersed in the process of propagation and interaction. This means that the microscopic particles depicted by linear quantum mechanics are unstable because they can be dispersed easily in motion. However, are the microscopic particles in nonlinear quantum mechanics stable? How do we verify their stability? Usually, three types of stability of microscopic particles are considered in nonlinear quantum mechanics, (1) with respect to perturbation of the initial state, (2) with respect to a perturbation of the dynamic equation governing the system dynamics (structural stability); (3) with respect to minimal energy state under action of an externally applied field. In the first case, the problem has been investigated using various approaches (linear and nonlinear). In the linear approximation the stability problem is usually reduced to an eigenvalue problem of linearized equations. In the nonlinear case it is reduced to the study of Lyapunov inequalities. Study of structural stability of microscopic particles can be done in the framework of dynamic equations under different types of perturbations. Attempts have been made to construct a general perturbation theory of microscopic particles based on the dynamic equations by using the Green function method and spectral transformation (i.e. a transformation from the configuration space (x, t) to the scattering data space based on the well-known two-time formalism). However, the latter is suitable only for a rather restricted class of perturbation functional. Although some results were obtained in this direction, the studies cannot be considered as complete. In some cases numerical studies are very effective tools. From the computational point of view, both problems can be investigated in the framework of a unified approach. In the first case one studies the dynamics, described by an unperturbed dynamic equation, of a perturbed or unperturbed initial state given in the form of a soliton solution, the stability of which is examined. In the second case, the initial state evolution is governed by a perturbed equation. In both cases the solutions depend on some parameters, which are slowly-varying-functions of time.

In the first case, a solution is considered as stable if initial perturbations are not magnified as the initial state evolves with time. In accordance with this definition, weakly radiating soliton-like solutions, which are not destroyed under initial perturbations, can be considered stable. Obviously, structure-stable solutions are the solutions that conserve their shape for a sufficiently long time. The notion of "sufficiently long time" is relative to the time scale of physical processes occurring in the systems. These problems are not studied further here. More details on these problems can be found in Makhankov et al.'s relevant publications[45-48].

However, how can the stability of macroscopic particles exposed in an externally applied field be proved? In this condition the interactions between the microscopic particles are very complicated, thus it is difficult to define and determine the behaviour and stability of each one individually using again the above strategies of initial and structure stability as well as the collision rule of the particles. Instead, we apply the fundamental work- energy theorem in classical physics: a mechanical system in the state of minimal energy is said to be stable because an external energy must be supplied, in order to change this state. Pang[42-43] demonstrated the stability of the microscopic particles described by nonlinear Schrödinger equation (8) using the minimum energy theorem. In fact, this stability principle is very effective, when microscopic particles are exposed to an externally applied field. This method is outlined in the following by Pang[42-43]

Let $\phi(x,t)$ represent the field of the particle, and assume that it has derivatives of all orders, and all integrations, and is convergent and finite. The Lagrange and Hamiltonian density function corresponding to the nonlinear Schrödinger equation (8) are represented by Eqs. (9) and (10), respectively, which involve all the nonlinear interactional energy, b $(\phi\phi^*)^2$, which can obstruct and suppress the dispersive effect of kinetic energy of microscopic particles. In the general case, the total energy of the particles is a function of t' and is represented by

$$E(t') = \int_{-\infty}^{\infty}\left[\left|\frac{\partial\phi}{\partial x'}\right|^2 - \frac{b}{2}\left|\phi\phi^*\right|^2 + V(x')|\phi|^2\right]dx' \tag{57}$$

However, in this case, b and $V(x')$ are not functions of t'. So, the total energy of the systems is a conservative quantity, i.e., $E(t')$=E=const.. We can demonstrate[42-43] that when $x' \to \pm\infty$, the solutions of Eq. (8) and $\phi(x',t')$ should tend to zero rapidly, i.e.,

$$\underset{|x'|\to\infty}{Lim}\,\phi(x',t') = \underset{|x|\to\infty}{Lim}\,\frac{\partial\phi}{\partial x'} = 0$$

then

$$\int_{-\infty}^{\infty}\phi^*\phi dx' = \int_{-\infty}^{\infty}\rho(x')dx' = \text{constant, or } \frac{\partial}{\partial t'}\int_{-\infty}^{\infty}\phi^*\phi dx' = 0$$

which is the mass conservation of the microscopic particle. Therefore, $\phi^*\phi dx' = \rho(x')dx'$ can be regarded as the mass in the interval of x' to x'+dx'. Thus the position of mass centre of a microscopic particle at $x_0' + v_e t'$ in nonlinear quantum mechanics can be represented by

$$\langle x'\rangle = x_g' = x_0 = \int_{-\infty}^{\infty}\phi'^* x'\phi dx' / \int_{-\infty}^{\infty}\phi^*\phi dx' \tag{58}$$

since

$$v = v_g = \frac{d}{dt'}\{\int_{-\infty}^{\infty}\phi^* x'\phi dx' / \int_{-\infty}^{\infty}\phi^*\phi dx'\} = \int_{-\infty}^{\infty}(\phi_{t'}^* x'\phi + \phi^* x'\phi_{t'})dx' / \int_{-\infty}^{\infty}\phi^*\phi dx'$$

(59)

then from Eq. (11) and its conjugate equation we can get that the velocity of mass centre of microscopic particle can be denoted by

$$v_g = d\langle x'\rangle / dt' = -2i\int_{-\infty}^{\infty}\phi^*\phi_{x'}dx' / \int_{-\infty}^{\infty}\phi^*\phi dx' \tag{60}$$

However, for different solutions of the same nonlinear Schrödinger equation (11), $\int_{-\infty}^{\infty}\phi^*\phi dx'$, $\langle x'\rangle$ and dx'/dt' can have different values. Therefore, it is unreasonable to compare the energy between a definite solution and other solutions. We should compare the energy of one particular solution to that of another solution. The comparison is only meaningful for many microscopic particle systems that have the same values of $\int_{-\infty}^{\infty}\phi^*\phi dx' = K$, $\langle x'\rangle = u$ and $d<x'>/dt' = \dot{u}$ at the same time t_0'. Based on these, we can finally determine the stability of the solutions of Eq. (11), for example, Eq. (21). Thus, we assume that the different solutions of Eq. (11) satisfy the following boundary conditions at a definite time t_0':

$$\int_{-\infty}^{\infty}\phi^*\phi dx' = K\,,\ \langle x'\rangle\big|_{t'=t_0'} = u(t_0'),\ \left.\frac{d<x'>}{dt'}\right|_{t'=t_0'} = \dot{u}(t_0') \tag{61}$$

Pang[42-43] assume the solution of Eq. (11) to have the form of Eq. (12). Substituting Eq. (12) into Eq. (57), we obtain the energy formula:

$$E = \int_{-\infty}^{\infty}[(\frac{\partial\varphi}{\partial x'})^2 + \varphi^2(\frac{\partial\theta}{\partial x'})^2 - \frac{b}{2}\varphi^4 + V(x')\varphi^2]dx' \tag{62}$$

At the same time, the equation (61) becomes

$$\int_{-\infty}^{\infty}\varphi^2 dx' = K\,,\quad \frac{\int_{-\infty}^{\infty} x'\varphi^2 dx'}{\int_{-\infty}^{\infty}\varphi^2 dx'} = u\left(t_0'\right)\,,\quad \frac{2\int_{-\infty}^{\infty}\varphi^2\frac{\partial\theta}{\partial x'}dx'}{\int_{-\infty}^{\infty}\varphi^2 dx'} = \dot{u}(t_0') \tag{63}$$

Finding the extreme value of the functional Eq. (62) under the boundary conditions of Eq. (63) by means of the Lagrange uncertain factor method, we obtain the following Euler equations:

$$\frac{\partial^2\varphi}{\partial x'^2} = \{V(x') + C_1(t_0')C_2(t_0')[x' - u(t_0')] + C_3(t_0')[2\frac{\partial\theta}{\partial t'} - \dot{u}(t_0')] + (\frac{\partial\theta}{\partial x'})^2\}\varphi - b\varphi^3 = \varphi \tag{64}$$

$$\varphi^2\frac{\partial^2\theta}{\partial x'^2} + 2\frac{\partial\varphi}{\partial x'}\varphi\frac{\partial\theta}{\partial x'} + 2C_3(t_0')\varphi\frac{\partial\varphi}{\partial t'} = 0 \tag{65}$$

where the Lagrange factors C_1 ,C_2 and C_3 are all functions of t' . Now, let $C_3(t_0') = -\frac{1}{2}\dot{u}(t_0')$.

If $$2\frac{\partial\theta}{\partial x'} - \dot{u}(t_0') \neq 0$$

we can get from Eq. (65)

$$\frac{2}{\varphi}\frac{\partial\varphi}{\partial x'} = \frac{-\partial^2\theta/\partial x'^2}{(\partial\theta/\partial x' - \dot{u}(t_0')/2)}$$

Integration of the above equation yields

$$\varphi^2 = \frac{g(t_0')}{(\partial\theta/\partial x' - \dot{u}(t_0')/2)} \quad \text{or} \quad \left.\frac{\partial\theta}{\partial x'}\right|_{t'=t_0'} = \frac{g(t_0')}{\varphi^2} + \frac{\dot{u}(t_0')}{2} \tag{66}$$

where $g(t_0')$ is an integral constant. Thus,

$$\theta(x',t') = g(t_0')\int_0^{x'}\frac{dx''}{\varphi^2} + \frac{\dot{u}(t_0')}{2}x' + M(t_0')\ , \tag{67}$$

here, $M(t_0')$ is also an integral constant. Again let

$$C_2(t_0') = \frac{1}{2}\ddot{u}(t_0')\ , \tag{68}$$

substituting Eqs. (65)-(68) into Eq. (64), we obtain

$$\frac{\partial^2\varphi}{\partial(x')^2} = \left\{V(x') + \frac{\ddot{u}(t_0')}{2}x' + \left[C_1(t_0') - \frac{\ddot{u}(t_0')}{2}u(t_0') + \frac{u^2(t_0')}{4}\right]\right\}\varphi - b\varphi^3 + \frac{g^2(t_0')}{\varphi^3} \tag{69}$$

Letting

$$C_1(t_0') = \frac{u(t_0')\ddot{u}(t_0')}{2} - \frac{\dot{u}^2(t_0')}{2} + M(t_0') + \beta' \tag{70}$$

where β' is an undetermined constant, which is a function of t' -independent, and assuming $Z = x' - u(t_0')$, then $\frac{\partial^2\varphi}{\partial(x')^2} = \frac{\partial^2\varphi}{\partial Z^2}$ is only a function of Z. To make the right-hand side of Eq. (69) be also a function of Z, the coefficients of φ, φ^3 and $1/\varphi^3$ must also be functions of Z, thus, $g(t_0') = g_0 = const$, and

$$V(x') + \frac{\ddot{u}(t_0')}{2}x' + M(t_0') - \frac{u^2(t)}{4} = \tilde{V}_0(Z)$$

Then, equation (69) becomes

$$\frac{\partial^2\varphi}{\partial(x')^2} = \{\tilde{V}[x' - u(t_0')] + \beta'\}\varphi - b\varphi^3 + \frac{g^2(t_0')}{\varphi^3} \tag{71}$$

Since $\tilde{V}(Z) = \tilde{V}_0[x' - u(t_0')] = 0$ in the present case. Hence, equation (71) becomes

$$\frac{\partial^2\varphi}{\partial(x')^2} = \beta'\varphi - b\varphi^3 + \frac{g^2(t_0')}{\varphi^3} \tag{72}$$

Therefore, φ is the solution of Eq. (72) for the parameters β'=constant and $g(t_0')$=constant. For sufficiently large $|Z|$ we may assume that $|\varphi| \le \tilde{\beta}/|Z|^{1+\Delta}$, where Δ is a small constant. However, in Eq. (72) we can only retain the solution $\varphi(Z)$ corresponding to $g(t_0')$ to ensure that $Lim_{|\varepsilon|\to\infty} d^2\varphi/dZ^2 = 0$. Thus, equation (72) becomes

$$\frac{\partial^2\varphi}{\partial(x')^2} = \beta'\varphi - b\varphi^3 \tag{73}$$

In fact, if $\partial\theta/\partial t' = \dot{u}/2$, then from Eqs.(71)-(72) we can verify that the solution given in Eq.(21) satisfies Eq. (73). In such a case, it is not difficult to show that the energy corresponding to the solution Eq. (21) of Eq. (73) has a minimal value under the boundary conditions given in Eq. (62). Thus, we can conclude that the solution of nonlinear Schrödinger equation (11), or microscopic particles with the wave-corpuscle duality in nonlinear quantum mechanics is stable in such a case.

3.4. The Classical Features of Motion of Microscopic Particles

3.4.1. The Feature of Classical Motion of Microscopic Particles

Since the microscopic particle described by the nonlinear Schrödinger equation (8) has a corpuscle feature and is also quite stable as mentioned above. Thus its motion in action of a potential field in space-time should have itself rules of motion. Pang[49-51] studied deeply this rule of motion of microscopic particles in such a case.

We know that the solution Eq. (21), shown in Fig.1, of the nonlinear Schrödinger equation (5) with different potentials at $A(\phi) = 0$, has the behavior of $\frac{\partial\phi}{\partial x'} = 0$ at $x' = x_0'$. Thus we can infer that the position of the soliton is localized at $x' = x_0'$ at t'=0, which is just the position of the mass center of microscopic particle, and is defined by Eq. (58). Its velocity is represented in Eq. (59). We now determine the acceleration of the mass center of the microscopic particles and its rules of motion in an externally applied potential.

Now utilizing Eq. (8) and its conjugate equation as follows

$$-i\hbar\frac{\partial\phi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\phi^* \pm b|\phi|^2\phi^* + V(\vec{r},t)\phi^*, \tag{74}$$

we can obtain

$$\frac{d}{dt'}\int_{-\infty}^{\infty}\phi^*\phi_{x'}dx' = \int_{-\infty}^{\infty}\phi_{t'}^*\phi_{x'}dx' + \int_{-\infty}^{\infty}\phi^*(\phi_{t'})_{x'}dx' = i\int_{-\infty}^{\infty}\{\phi^*\frac{\partial}{\partial x'}[\phi_{x'x'} + b\phi^*\phi^2$$

$$-V\phi] - [\phi_{x'x'}^* - b\phi(\phi^*)^2 - V\phi^*]\phi_{x'}\}dx' = i\int_{-\infty}^{\infty}\phi^*\frac{\partial V}{\partial x'}\phi dx' \quad (75)$$

where $x' = x/\sqrt{\hbar^2/2m}$, $t' = t/\hbar$. We here utilize the following relations and the boundary conditions:

$\int_{-\infty}^{\infty}(\phi^*\phi_{x'x'x'} - \phi_{x'x'}^*\phi_{x'})dx' = 0, \int_{-\infty}^{\infty}b(\phi^{*2}\phi\phi_{x'} + \phi^*\phi^2\phi_{x'}^*)dx' = 0$, $\int \phi^*\phi dx' =$ constant (or a function of t') and

$$\lim_{|x'|\to\infty}\phi^* x'\frac{\partial\phi}{\partial x'} = \lim_{|x'|\to\infty}\frac{\partial\phi^*}{\partial x'}x'\phi = 0, \quad \lim_{|x'|\to\infty}\phi(x',t') = \lim_{|x'|\to\infty}\phi_{x'}(x',t') = 0$$

where $\phi_{x'} = \frac{\partial\phi}{\partial x'}, \phi_{x'x'x'} = \frac{\partial^3\phi}{\partial x'^3}$. Thus, we can get

$$\frac{d}{dt'}\int_{-\infty}^{\infty}\phi^* x'\phi dx' = \int_{-\infty}^{\infty}(\frac{\partial\phi^*}{\partial t'}x'\phi + \phi^* x'(\frac{\partial\phi}{\partial t'})dx' = -2i\int_{-\infty}^{\infty}\phi^*\phi_{x'}dx' \quad (76)$$

In the systems, the position of mass centre of microscopic particle can be represented by Eq. (58), thus the velocity of mass centre of microscopic particle is represented by Eq. (59). Then, the acceleration of mass centre of microscopic particle can also be denoted by

$$\frac{d^2}{dt'^2} < x' > = -2i\frac{d}{dt'}\{\int_{-\infty}^{\infty}\phi^*\phi_{x'}dx' / \int_{-\infty}^{\infty}\phi^*\phi dx'\} = -2\int_{-\infty}^{\infty}\phi^* V_x\phi^* dx' = -2 < \frac{\partial V}{\partial x'} > \quad (77)$$

If ϕ is normalized, i.e., $\int_{-\infty}^{\infty}\phi^*\phi dx' = 1$, then the above conclusions also are not changed.

where V=V(x') in Eq. (77) is the external potential field experienced by the microscopic particles. We expand $\frac{\partial V}{\partial x'}$ at the mass centre $x' = < x' > = x_0'$ as

$$\frac{\partial V(x')}{\partial x'} = \frac{\partial V(< x' >)}{\partial < x' >} + (x' - < x' >)\frac{\partial^2 V(< x' >)}{\partial < x' >^2} + \frac{1}{2!}(x' - < x' >)^2\frac{\partial^3 V(< x' >)}{\partial < x' >^3} + \cdots$$

Taking the expectation value on the above equation, we can get

$$\left\langle\frac{\partial V(x')}{\partial x'}\right\rangle = \frac{\partial V(< x' >)}{\partial < x' >} + \frac{1}{2!} < (x' - < x' >)^2 > \frac{\partial^3 V(< x' >)}{\partial < x' >^3}$$

where

$$\Delta_{ij} = \left\langle((x' - \langle x'\rangle)^2\right\rangle = \left\langle(x_i' - x_0')(x_j' - x_0')\right\rangle = \left\langle(x_i' - \langle x_i'\rangle)(x_j' - \langle x_j'\rangle)\right\rangle$$

For the microscopic particle described by Eq. (8) or Eq. (11), the position of the mass center of the particle is known and determinant, which is just < x'>= x_0' =constant, or 0. Since we here study only the rule of motion of the mass centre x_0, which means that the terms containing x_0' in <x'^{2}> are considered and included, then $< (x' - < x' >)^2 >$ =0 can be obtained. Thus

$$\left\langle\frac{\partial V(x')}{\partial x'}\right\rangle = \frac{\partial V(< x' >)}{\partial < x' >}$$

Pang[49-53] finally obtained the acceleration of mass center of microscopic particle in the nonlinear quantum mechanics, Eq. (77), which is denoted as

$$\frac{d^2}{dt'^2} < x' > = -2\frac{\partial V(<x'>)}{\partial <x'>} \tag{78}$$

Returning to the original variables, the equation (78) becomes

$$m\frac{d^2x_0}{dt^2} = -\frac{\partial V}{\partial x_0} \tag{79}$$

where x_0' =<x'> is the position of the mass centre of microscopic particle. Equation (79) is the equation of motion of mass center of the microscopic particles in the nonlinear quantum mechanics. It resembles quite the Newton-type equation of motion of classical particles, which is a fundamental dynamics equation in classical physics. Thus it is not difficult to conclude that the microscopic particles depicted by the nonlinear quantum mechanics have a property of the classical particle.

The above equation of motion of particles can also derive from Eq.(8) by another method. As it is known, the momentum of the particle depicted by Eq.(8) is obtained from Eq.(9) and denoted by $P = \frac{\partial L}{\partial \dot\phi} = -i\int_{-\infty}^{\infty}\left(\phi^*\phi_{x'} - \phi_{x'}^*\phi\right)dx'$. Utilizing Eq.(8) and Eqs.(74)-(76) Pang obtained[25-27,49]

$$\frac{dP}{dt'} = \int_{-\infty}^{\infty} 2V(x')\frac{\partial}{\partial x'}|\phi|^2\, dx' = -2\int_{-\infty}^{\infty}\frac{\partial V}{\partial x'}|\phi|^2\, dx' = -2\left\langle\frac{\partial V(x')}{\partial x'}\right\rangle \tag{80}$$

where the boundary condition of $\phi(x') \to 0$ as $|x'| \to \infty$ is used. Utilizing again the above result of $\left\langle\frac{\partial V(x')}{\partial x'}\right\rangle = \frac{\partial V(<x'>)}{\partial <x'>}$ we can get also that the acceleration of the mass center of the particle is the form of

$$\frac{dP}{dt'} = -2\frac{\partial V(x_0')}{\partial x_0'} \quad \text{or} \quad m\frac{d^2x_0}{dt^2} = -\frac{\partial V}{\partial x_0} \tag{81}$$

where x_0' is the position of the center of the mass of the macroscopic particle. This is the same as Eq. (79) . Therefore, we can confirm that the microscopic particles in the nonlinear quantum mechanics satisfy the Newton-type equation of motion. for a classical particle.

3.4.2. Lagrangian and Hamilton Equations of Microscopic Particle

Using the above variables ϕ in Eq. (8) and ϕ^* in Eq. (74) one can determine the Poisson bracket and write further the equations of motion of microscopic particles in the form of Hamilton's equations. For Eq. (8), the variables ϕ and ϕ^* satisfy the Poisson bracket:

$$\left\{\phi^{(a)}(x), \phi^{(b)}(y)\right\} = i\delta^{ab}\delta(x-y) \tag{82}$$

where $$\{A, B\} = i\int_{-\infty}^{\infty}\left(\frac{\delta A}{\delta\phi}\frac{\delta B}{\delta\phi^*} - \frac{\delta B}{\delta\phi}\frac{\delta A}{\delta\phi^*}\right)$$

The corresponding Lagrangian density L in Eq. (9) associated with Eq. (8) can be written in terms of $\phi(x',t')$ and its conjugate ϕ^* (x',t') viewed as independent variables. From Eq. (9) Pang[25-27,49-51]can find out

$$\frac{\partial L'}{\partial\phi^*} = \frac{i\hbar}{2}\phi_t - V(x)\phi + b(\phi^*\phi)\phi \quad \text{and} \quad \partial_t\left(\frac{\partial L'}{\partial\phi_t^*}\right) + \nabla.\left(\frac{\partial L'}{\partial\nabla\phi^*}\right) = -\frac{i\hbar}{2}\phi_t - \frac{\hbar^2}{2m}\nabla.\nabla\phi$$

where L' =L, thus we can obtain

$$\frac{\partial L'}{\partial\phi^*}-\partial_t(\frac{\partial L'}{\partial\phi_t^*})-\nabla.(\frac{\partial L'}{\partial\nabla\phi^*})=i\hbar\phi_t+\frac{\hbar^2}{2m}\nabla^2\phi-V(x)\phi+b(\phi^*\phi)\phi \quad (83)$$

Through comparison with Eq. (5) at $A(\phi)=0$ Pang gets

$$\frac{\partial L'}{\partial\phi^*}=\partial_t(\frac{\partial L'}{\partial\phi_t^*})+\nabla.(\frac{\partial L'}{\partial\nabla\phi^*}) \quad \text{or} \quad \frac{\partial L'}{\partial\phi}=\partial_t(\frac{\partial L'}{\partial\phi_t})+\nabla.(\frac{\partial L'}{\partial\nabla\phi}) \quad (84)$$

Equation (84) is just the well-known Euler-Lagrange equation for this system. This show that the nonlinear *Schrödinger* equation amount to Euler-Lagrange equation in nonlinear quantum mechanics, in other word, the dynamic equation, or the nonlinear Schrödinger equation can be obtained from the Euler-Lagrange equation in nonlinear quantum mechanics, if the Lagrangian function of the system is known. This is different from quantum mechanics, in which the dynamic equation is the linear Schrödinger equation, instead of the Euler-Lagrange equation.

On the other hand, Pang[25-27,49-51] got also the Hamilton equation of the microscopic particle from the Hamiltonian density of this system in Eq. (10). In fact, we can obtain from Eq. (10)

$$\frac{\delta H'}{\delta\phi^*}=-\frac{\hbar^2}{2m}\nabla^2\phi+V(x)\phi-b(\phi^*\phi)\phi \quad (85)$$

Where H' =H. Then from Eq. (10) we can give

$$i\hbar\frac{\partial\phi}{\partial t}=\frac{\delta H'}{\delta\phi^*}=-\frac{\hbar^2}{2m}\nabla^2\phi+V(x)\phi-b(\phi^*\phi)\phi$$

Thus $$i\hbar\frac{\partial\phi}{\partial t}=\frac{\delta H'}{\delta\phi^*}, \text{or } i\hbar\frac{\partial\phi^*}{\partial t}=-\frac{\delta H'}{\delta\phi} \quad (86)$$

Equation (86) is just the complex form of Hamilton equation in nonlinear quantum mechanics. In fact, the Hamilton equation can be also represent by the canonical coordinate and momentum of the particle. In this case the canonical coordinate and momentum are defined by

$$q_1=\frac{1}{2}\left(\phi+\phi^*\right), \quad p_1=\frac{\partial L'}{\partial\left(\partial_{t'}q_1\right)}; q_2=\frac{1}{2i}\left(\phi-\phi^*\right), \quad p_2=\frac{\partial L'}{\partial\left(\partial_{t'}q_2\right)}$$

Thus, the Hamiltonian density of the system H in Eq. (10) takes the form

$$H'=\sum_i p_i\partial_{t'}q_i-L'$$

where H' = H , and the corresponding variation of the Lagrangian density L'= L can be written as

$$\delta L'=\sum_i\frac{\delta L'}{\delta q_i}\delta q_i+\frac{\delta L'}{\delta\left(\nabla q_i\right)}\delta\left(\nabla q_i\right)+\frac{\delta L'}{\delta\left(\partial_{t'}q_i\right)}\delta\left(\partial_{t'}q_i\right) \quad (87)$$

From Eq. (87), the definition of p_i, and the Euler-Lagrange equation,

$$\frac{\partial L'}{\partial q_i}=\nabla.\frac{\partial L'}{\partial\nabla q_i}+\frac{\partial p_i}{\partial t}$$

one obtains the variation of the Hamiltonian in the form of

$$\delta\mathcal{H}'=\sum_i\int\left(\partial_{t'}q_i\delta p_i-\partial_{t'}p_i\delta q_i\right)dx'$$

Thus, one pair of dynamic equations can be obtained and expressed by

$$\frac{\partial q_i}{\partial t'}=\delta H'/\delta p_i, \quad \frac{\partial p_i}{\partial t'}=-\delta H'/\delta q_i \quad (88)$$

This is analogous to the Hamilton equation in classical mechanics and has same physical significance with Eqs.(86), but the latter is used often in nonlinear quantum mechanics. This result shows that the nonlinear *Schrödinger* equation describing the dynamics of microscopic particle can be obtained from the classical Hamilton equation in the case, if the Hamiltonian of the system is known. Obviously, such methods of finding dynamic equations are impossible in the quantum mechanics. As it is known, the Euler-Lagrange equation and Hamilton equation are fundamental equations in classical theoretical (analytic) mechanics, and were used to describe laws of motions of classical particles. This means that the microscopic particles possess evidently classical features in nonlinear quantum mechanics. From this study we seek also a new way of finding the equation of motion of the microscopic particles in nonlinear systems, i.e., only if the Lagrangian or Hamiltonian of the system is known, we can obtain the equation of motion of microscopic particles from the Euler-Lagrange or Hamilton equations.

On the other hand, from de Broglie relation $E = h\upsilon = \hbar\omega$ and $\vec{p} = \hbar\vec{k}$, which represent the wave-corpuscle duality of the microscopic particles in quantum theory, we see that the frequency ω and wavevector $\vec{k}$ can play the roles as the Hamiltonian of the system and momentum of the particle, respectively, even in the nonlinear systems and has thus the relation:

$$\frac{d\omega}{dt'} = \frac{\partial\omega}{\partial k}\bigg|_{x'}\frac{dk}{dt'} + \frac{\partial\omega}{\partial x'}\bigg|_{k}\frac{\partial x'}{\partial t'} = 0$$

as in the usual stationary media. From the above result we also know that the usual Hamilton equation in Eq. (85) for the nonlinear systems remain valid for the microscopic particles. Thus, the Hamilton equation in Eq. (85) can be now represented by another form[25-27,52-53]:

$$\frac{dk}{dt'} = -\frac{\partial\omega}{\partial x'}\Big|_{k} \quad \text{and} \quad \frac{dx'}{dt'} = \frac{\partial\omega}{\partial k}\Big|_{x'} \tag{89}$$

in the energy picture, where $k = \partial\theta/\partial x'$ is the time-dependent wavenumber of the microscopic particle, $\omega = -\partial\theta/\partial t'$ is its frequency ,θ is the phase of the wave function of the microscopic particles.

3.4.3. Confirmation of Correctness of the above Conclusions

We now use some concrete examples to confirm the correctness of the laws of motion of the microscopic particles mentioned above in the nonlinear quantum mechanics[25-27,49-51]:.

(1) For the microscopic particles described by Eq. (8) $V = 0$ and constant, of which the solutions are Eq. (20) or (21) and (41), respectively, we obtain that the acceleration of the mass

centre of microscopic particle is zero because of $m\frac{d^2}{dt^2}\langle x\rangle = -\frac{\partial V(<x>)}{\partial <x>}$ =0 in this case. This means that the velocity of the particle is a constant. In fact, if inserting Eq. (20) into Eq. (79) we can obtain the group velocity of the particle v_g= $d<x'>/dt'$= $-2i\int_{-\infty}^{\infty}\phi^*\phi_{x'}dx'$= $v_e = -4\xi$ = constant. This manifests that the microscopic particle moves in uniform velocity in space-time, its velocity is just the group velocity of the soliton, thus the energy and momentum of the microscopic particles can retain in the motion process. These properties are the same with classical particle.

On the other hand, If the dynamic equation (89) is used we can obtain from Eq. (20) that the acceleration and velocity of the microscopic particle are

$$\frac{dk}{dt'} = 0 \quad \text{and} \quad v_g = \frac{dx'}{dt'} = \frac{\partial\omega}{\partial k} = v_e = -4\xi\,,$$

respectively, where

$$\omega = -\partial\theta/\partial t' = 4(\xi^2 - \eta^2) = k^2 - 4\eta^2\,,\ k = \partial\theta/\partial x' = -2\xi\,,\ \theta = -4\left(\xi^2 - \eta^2\right)t' - 2\xi x' + \theta_0$$

For the solution in Eq. (41) at V =constant, $\theta = v_e\left(x' - x_0'\right)/2 - (\beta - v_e^2/4 - C)t'$, then $\omega =$ $(\beta - v_e^2/4 - C), k = v_e/2$, thus $\frac{dk}{dt'} = 0$ and $v_g = \frac{dx'}{dt'} = \frac{\partial\omega}{\partial k} = -2k = -v_e$.

These results of the acceleration and velocity of microscopic particle are same with the above data obtained from Eqs.(20) and (79). This indicates that these moved laws shown in Eqs.(78), (79) , (81), (86), (88), and (89) are self-consistent, correct and true in nonlinear quantum mechanics.

(2) For the case of $V(x') = \alpha x'$, the solution of Eq. (8) is Eq. (44) by Chen and Liu[40-41], which is also composed of a envelope and carrier wave. The mass centre of the particle is at x'_0, which is its localized position. From Eq. (79) we can determined the accelerations of the mass center of the microscopic particle in this case, which is given by

$$\frac{d^2 x_0'}{dt'^2} = -2\frac{\partial V(\langle x'\rangle)}{\partial\langle x'\rangle} = -2\alpha = \text{constant} \tag{90}$$

On the other hand, from Eq. (44) we know that

$$\theta = 2(\xi - \alpha t')x' + \frac{4\alpha^2 t'^3}{3} - 4\alpha\xi t'^2 + 4(\xi^2 - \eta^2)t' + \theta_0, \tag{91}$$

where ξ is same with ξ' in Eq. (44). Utilizing again Eq. (89) we can find

$$k = 2(\xi - \alpha t')$$

$$\omega = 2\alpha x' - 4(\xi - \alpha t')^2 + (2\eta)^2 = 2\alpha x' - k^2 + (2\eta)^2$$

Thus, the group velocity of the microscopic particle is found out from

$$v_g = \frac{d\overline{x'}}{dt'} = \frac{\partial\omega}{\partial k} = 4(\xi - \alpha t') \tag{92}$$

Then its acceleration is given by

$$\frac{d^2\tilde{x}'}{dt'^2} = \frac{dk}{dt'} = -2a = \text{constant}, \quad \text{here}(x_0' = \tilde{x}') \tag{93}$$

Comparing Eq. (90) with Eq. (91) we find that they are also same. This indicates that Eqs. (79), (81), (86) , (88) and (89) are correct. In such a case the microscopic particle moves in an uniform acceleration. This is similar with that of classical particle in an electric field.

(3) For the case of $V(x') = \alpha^2 x'^2$, which is a harmonic potential, the solution of Eq. (8) in this case is Eq. (47) obtained by Chen and Liu[40-41]. This solution contain also a envelop and carrier wave, and has also a mass centre, its position is at x'_0, which is the position of the microscopic particle. When Eq. (79) is used to determine the properties of motion of the particle in this case Pang [27-30]found out that the accelerations of the center of mass of the particle is

$$\frac{d^2 x_0'}{dt'^2} = -4\alpha^2 x_0' \tag{94}$$

At the same time, from Eq. (47) we gain that

$$\theta = 2\xi(x' - x_0')\cos[2\alpha(t' - t_0')] + 4\eta^2(t' - t_0') - (\xi^2/\alpha)\sin[4\alpha(t' - t_0')] + \theta_0 \tag{95}$$

where ξ is same with ξ' in Eq. (47). From Eqs.(89) and (95) we can find

$$k = 2\xi\cos 2\alpha(t' - t_0'),$$

$$\omega = 4a\xi x'\sin 2\alpha(t' - t_0') - 4\xi^2\cos 4\alpha(t' - t_0') - 4\eta^2$$

$$= 2\alpha x'(4\xi^2 - k^2)^{1/2} - 2k^2 + 4(\xi^2 - \eta^2),$$

Thus, the group velocity of the microscopic particle is

$$v_g = \left.\frac{\partial\omega}{\partial k}\right|_{x'} = \frac{\alpha x'}{\xi}\frac{k}{\sqrt{1 - k^2/4\xi^2}} - 2k = 2\alpha x'\,\text{ctg}\left[2\alpha(t' - t_0')\right] - 4\xi\cos\left[2\alpha(t' - t_0')\right],$$

While its acceleration is

$$\frac{dk}{dt'} = -\left.\frac{\partial \omega}{\partial x'}\right|_k = -2\alpha\sqrt{4\xi^2 - k^2} = -4\xi\alpha \sin\left[2\alpha\left(t' - t_0'\right)\right]. \tag{96}$$

Since $\frac{d^2\tilde{x}'}{dt'^2} = \frac{dk}{dt'}$, here $\left(\tilde{x}' = x_0'\right)$, we have

$$\frac{dk}{dt'} = \frac{d^2\overline{x'}}{dt'^2} = -4\xi\alpha \sin[2\alpha(t' - t_0')]$$

and

$$x' = \frac{\xi}{\alpha}\sin[2\alpha(t - t_0')] \tag{97}$$

Finally, the acceleration of the microscopic particle is

$$\frac{d^2\overline{x'}}{dt'^2} = \frac{dk}{dt'} = -4\alpha^2\overline{x}' \tag{98}$$

Equation (98) are also the same with Eq. (94). Thus we confirm also the validity of Eqs.(76), (79), (81), (84), (86) and (88)- (89). In such a case the microscopic particle moves in harmonic form. This resembles also with the result of motion of classical particle.

From the above studies we draw the following conclusions[25-27,51-53]. (1) The motions of microscopic particles in the nonlinear quantum mechanics can be described by not only the nonlinear Schrödinger equation but also Hamiltonian principle, Lagrangian and Hamilton equations, its changes of position with changing time satisfy the law of motion of classical particle in both uniform and inhomogeneous. This not only manifests that the natures of microscopic particles described by nonlinear quantum mechanics differ completely from those in the linear quantum mechanics but displays sufficiently the corpuscle nature of the microscopic particles.

(2) The external potentials can change the states of motion of the microscopic particles, although it cannot vary its wave-corpuscle duality, for example, the particle moves with a uniform velocity at V(x')=0 or constant , or in an uniform acceleration at $V\left(x'\right) = ax'$, which corresponds to the motion of a charge particle in a uniform electric field, but when $V\left(x'\right) = \alpha^2 x'^2$ the macroscopic particle performs localized vibration with a frequency of 2α and an amplitude of ξ/α, the corresponding classical vibrational equation is $x' = x_0' \sin\omega t'$,with $\omega = 2a$ and $x_0' = \xi/\alpha$. The laws of motion of the center of mass of microscopic particles expressed by Eq. (79) and Eqs. (86) – (89) in the nonlinear quantum mechanics are consistent with the equations of motion of the macroscopic particles.

The correspondence between a microscopic particle and a macroscopic object shows that microscopic particles described by the nonlinear quantum mechanics have exactly the same moved laws and properties as classical particles. These results not only verify the necessity of development and correctness of the nonlinear quantum mechanics, but also exhibit clearly the limits and approximation of the linear quantum mechanics and can solve these difficulties of the linear quantum mechanics and problems of contention in it as described in Introduction. Therefore, the results mentioned above have important significances in physics and nonlinear science.

3.5. The General Conservation Laws of Motion of Particles

It is well known that classical particles satisfy the invariance and conservation laws of mass, energy and momentum and angular momentum, which are the elementary and universal laws of matter in nature. In the meanwhile, we know from Eq. (22) that the microscopic particles have a determinant mass, momentum and energy in the nonlinear quantum mechanics. However, whether the mass, momentum and energy of the microscopic particles have also such a invariance and conservation in the nonlinear quantum mechanics. In this section we will study this problem and give further the conservation laws of mass, energy and momentum and angular momentum, etc., of the microscopic particles depicted by the nonlinear Schrödinger equation (5). We will find that the microscopic particles satisfy also these conventional conservation laws and have the properties of classical particles.

In the nonlinear quantum mechanics from nonlinear Schrödinger equation (8) we can define the number density, number current, densities of momentum and energy for the microscopic particle as[23-27]

$$\rho = |\phi|^2,\ p = -i\hbar(\phi^*\phi_x - \phi\phi_x^*)$$

$$J = i\hbar(\phi^*\phi_x - \phi\phi_x^*), \in = \frac{\hbar^2}{2m}|\phi_x|^2 - \frac{b}{2}|\phi\phi^*|^2 + V(x)|\phi|^2 \tag{99}$$

where. $\phi_x = \frac{\partial}{\partial x}\phi(x,t), \phi_t = \frac{\partial}{\partial t}\phi(x,t)$. From Eq.(8) and its conjugate equation (74) as well as Eqs.(9)-(10) and (99) we can obtain[23-37]

$$\frac{\partial p}{\partial t'} = \frac{\partial}{\partial x'}[2(\frac{\partial \phi}{\partial x'})^2 + b|\phi^*\phi|^2 + 2V|\phi|^2 - (\phi^*\frac{\partial^2\phi}{\partial x'^2} + \phi\frac{\partial^2\phi^*}{\partial x'^2}) + 2iV(\phi^*\frac{\partial\phi}{\partial x'} - \phi\frac{\partial\phi^*}{\partial x'})]$$

$$\frac{\partial \rho}{\partial t'} = \frac{\partial J}{\partial x'}, \frac{\partial \in}{\partial x'} = \frac{\partial}{\partial x'}[\rho p + i(\frac{\partial\phi^*}{\partial x'}\frac{\partial^2\phi}{\partial x'^2} - \frac{\partial\phi}{\partial x'}\frac{\partial^2\phi^*}{\partial x'^2}) - iV(\phi^*\frac{\partial\phi}{\partial x'} - \phi\frac{\partial\phi^*}{\partial x'})]$$

where $x' = x\sqrt{2m}/\hbar, t' = t/\hbar$ .Thus, we get the following forms for the integral of motion

$$\frac{\partial}{\partial t'}M = \frac{\partial}{\partial t'}\int \rho dx' = 0, \frac{\partial}{\partial t'}P = \frac{\partial}{\partial t'}\int p dx' = 0, \frac{\partial E}{\partial t'} = \frac{\partial}{\partial t'}\int \in dx' = 0, \tag{100}$$

These formulae represent just the conservation of mass, momentum and energy in such a case. This shows that the mass, momentum and energy of the microscopic particles in the nonlinear quantum mechanics still satisfy conventional rules of conservation of matter. We can easily inspect that the mass, momentum and energy of the particles described by Eq. (11) or Eq. (19) satisfy these conservation rules shown in Eq. (100). These results not only indicate that the microscopic particles in the nonlinear quantum mechanics possess a corpuscle feature but also manifests that the nonlinear quantum mechanics can give the common rules of motions of matter in nature, thus we have the reasons to believe that the new theory is correct.

We understand clearly from Eqs.(99)-(100) the really physical significance of wave function $\phi(x',t')$ in Eq. (12) in the nonlinear quantum mechanics. I represents in truth the states and properties of microscopic particles, the $|\phi(x',t')|^2$ represents the number or mass density of particles at certain point in place-time, its integration with respect to x over whole splace represents the effective mass of the microscopic particle. Therefore, $|\phi(x',t')|^2$ represents no longer the probability of the particle occurred at certain point in the linear quantum mechanics. This is just the true essence and meaning of the wave function of the particles in Eq. (12) in nonlinear quantum mechanics.

As matter of fact, the microscopic particles described by the nonlinear Schrödinger equation (8) have much of conservation laws, except for the conservations shown in Eqs. (99)—(100). Sulem[54] obtained other conservation laws by the Noether theorem, such as, the invariance under space rotation and conservation law of angular momentum $\vec{M} = i\int \vec{x}'\times\left(\phi^*\nabla\phi - \phi\nabla\phi^*\right)dx'$ and Galilean invariance, in the latter the action of the system and the nonlinear Schrödinger equation (8) are invariant under the Galilean transformation[54]:

$$\begin{gathered} x' \to x'' = x' - vt', t' \to t'' = t' \\ \phi(x',t') \to \phi''(x'',t'') = \exp[-i(vx''/2 + \vec{v}.\vec{v}t''/2)]\phi(x'',t'') \end{gathered} \tag{101}$$

where $\delta \vec{x}' = -\upsilon t', \delta t' = 0$ and $\delta\hat{\phi} = \phi''(x'',t'') - \phi(x',t') = -(i/2)\upsilon x'\phi(x',t')$

3.6. Classical Natures of Collision of Microscopic Particles with Attractive Nonlinear Interactions

As a matter of fact, the properties of collision of solitons obtained from different nonlinear Schrödinger equations were in detail studied, such as Zakharov and Shabat[35-36] studied firstly and discussed analytically the properties of collision of two particles depicted by the nonlinear Schrödinger equation (11) at b=1>0 and b< 0 using Zakharov and Shabat equation[35-36, 55]:

$$\psi_{1x'} - q\psi_2 = -i\lambda\psi_1 \tag{102}$$

$$\psi_{2x'} + q^*\psi_1 = i\lambda\psi_2 \tag{103}$$

where

$$q = \frac{i\phi}{(1-s^2)^{1/2}} = i(\frac{b}{2})^{1/2}\phi\,, \quad \lambda = \bar{k}s \tag{104}$$

and the inverse scattering method. They[35-36] find from calculation that the mass centre and phase of particle occur only change after this collision for Eq.(11) at b=1>0 . The translations of the mass centre $x_{0m}^{'+}$ and phase θ_m^+ of m[th] particles, which moves to a positive direction after this collision, can be represented, respectively, by

$$x_{0m}^{'+} - x_{0m}^{'} = \frac{1}{\eta_m}\prod_{p=m+1}^{N}\left|\frac{\lambda_m-\lambda_p}{\lambda_m-\lambda_p^*}\right| < 0, \text{ and } \theta_m^+ - \theta_m = -2\prod_{p=m+1}^{N}\arg\left(\frac{\lambda_m-\lambda_p}{\lambda_m-\lambda_p^*}\right) \tag{105}$$

where η_m and λ_m are some constants related to the amplitude and eigenvalue of m[th] particles, respectively. The equations show that shift of position of mass centre of the particles and their variation of phase are a constants after the collision of two particles moving with different velocities and amplitudes. The collision process of the two particles can be described from Eq. (105) as follows. Before the collision and in the case of $t' \to -\infty$ the slowest soliton is in the front while the fastest at the rear, they collide with each other at t'=0, after the collision and $t' \to \infty$, they are separated and the positions just reversed. Thus Zakharov and Shabat[35-36] obtained that as the time t varies from $-\infty$ to ∞, the relative change of mass centre of two particles, $\Delta x_{0m}^{'}$, and their relative change of phases, $\Delta\theta_m$, can, respectively, denoted as

$$\Delta x_{0m}^{'} = x_{0m}^{'+} - x_{0m}^{'-} = \frac{1}{\eta_m}\left(\sum_{k=m+1}^{N}\ln\left|\frac{\lambda_m-\lambda_p}{\lambda_m-\lambda_p^*}\right| - \sum_{k=1}^{m-1}\ln\left|\frac{\lambda_m-\lambda_p}{\lambda_m-\lambda_p^*}\right|\right) \tag{106}$$

and

$$\Delta\theta_m = \theta_m^+ - \theta_m^- = 2\prod_{k=1}^{m-1}\arg\left(\frac{\lambda_m-\lambda_p}{\lambda_m-\lambda_p^*}\right) - 2\prod_{k=m+1}^{N}\arg\left(\frac{\lambda_m-\lambda_p}{\lambda_m-\lambda_p^*}\right) \tag{107}$$

where $x_{0m}^{'-}$ and phase θ_m^- are the mass centre and phase of m[th] particles at inverse direction or initial position, respectively. Equation (106) can be interpreted by assuming that the microscopic particles collide pairwise and every microscopic particle collides with others. In each paired collision, the faster microscopic particle moves forward by an amount of $\eta_m^{-1}\ln\left|\left(\lambda_m-\lambda_k^*\right)/\lambda_m-\lambda_k\right|$, $\lambda_m > \lambda_k$, and the slower one shifts backwards by an amount of $\eta_k^{-1}\ln\left|\left(\lambda_m-\lambda_k^*\right)/\lambda_m-\lambda_k\right|$. The total shift is equal to the algebraic sum of their shifts during the paired collisions. So that there is no effect of multi-particle collisions at all. In other word, in the collision process in each time the faster particle moves forward by an amount of phase shift, and the slower one shifts backwards by an amount of phase. The total shift of the particles is equal to the algebraic sum of those of the pair during the paired collisions. The situation is the same with the phases. This rule of collision of the microscopic particles described by the nonlinear Schrödinger equation (11) is the same as that of classical particles, or speaking, meet also the collision law of macroscopic particles, i.e., during the collision these microscopic particles interact and exchange their positions in the space-time trajectory as if they had passed through each other. After the collision, the two microscopic particles may appear to be instantly translated in space and/or time but otherwise unaffected by their interaction. The translation is called a phase shift as mentioned above. In one dimension, this process results from two microscopic particles colliding head-on from opposite directions, or in one direction between two particles with different amplitudes or velocities. This is possible because the velocity of a particle depends on the amplitude. The two microscopic particles surviving a collision completely unscathed demonstrate clearly the corpuscle feature of the microscopic particles. This property separates the microscopic particles (solitons) described by the nonlinear quantum mechanics from the particles in the linear quantum mechanical regime. Thus this demonstrates the classical feature of the microscopic particles.

At the same time, Desem and Chu[56-57] pay attention to the features of the above solitons in collision process by Zakharov and Shabat's approach. Tan et al[58] carried out numerical simulation for the collision process between two soliton solutions of by the nonlinear Schrödinger equation (11) at b=1>0 using the Fourier pseudo-spectral method with 256 basis functions for the spatial discretization together with the fourth-order Runge-Kutta method for time-evolution.

Also, Zakharov and Shabat[35-36] discussed analytically the properties of collision of two particles depicted by the nonlinear Schrödinger equation (11) at b< 0 using Zakharov and Shabat equation[35-36, 55]. The result shows that the feature of collision of two solitons in this case are basically similar with the above properties. In the meanwhile, Aossey et al[59] investigated numerically the detailed structure, mechanism and rules of collision of the microscopic particles described by the nonlinear Schrödinger equation (11) at b<0 and obtained the rules of collision of the two solitons by a macroscopic model.

The above collision features of the microscopic particle are mainly obtained by using the inverse scattering method and Zakharov and Shabat equation in Eqs.(102)-(103). However, the properties of collisions of microscopic particles can be obtained by numerically solving Eq. (8). Numerical simulation can reveal more detailed feature of collision between two microscopic particles. Pang[25,51] studied numerically the features of collision of microscopic particles described by the nonlinear Schrödinger equation (8) at b>0 by fourth–order Runge-Kutta method[60]. For this purpose in one dimensional case we began by dividing Eq.(8) into the following two-equations

$$i\hbar\frac{\partial\phi}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2}=\chi\phi\frac{\partial u}{\partial x}, \tag{108}$$

$$M(\frac{\partial^2 u}{\partial t^2}-v_0\frac{\partial^2 u}{\partial x^2})=\chi\frac{\partial}{\partial x}\left(|\phi|^2\right). \tag{109}$$

Equations (108)-(109) may be thought to describe the features of motion of studied particle and another particle (such as, phonon) or background field (such as, lattice) with mass M and velocity v_0, u is the characteristic quantity of vibration (such as, displacement) of the background field (or phonon). The coupling between the two modes of motion is caused by the deformation of the background field through the studied particle–background field coupling, such as, dipole-dipole interaction, χ is the coupling coefficient between them and represents the change of interaction energy between the studied particle and background field due to an unit variation of the field. The relation between the two modes of motion due to their interaction can be represented by

$$\frac{\partial u}{\partial x}=\frac{\chi}{M(v^2-v_0^2)}|\phi|^2 \tag{110}$$

If inserting Eq.(110） into Eq.(108) yields just the nonlinear Schrödinger equation (8) at V(x)=constant, where $b=\dfrac{\chi^2}{M(v^2-v_0^2)}$ is a nonlinear coupling coefficient. This investigation shows clearly that the nonlinear interaction $b|\phi|^2\phi$ comes from the coupling interaction between two particles or the studied particle and background field, of which the real motion is considered, instead of replacement by an average field. Thus this confirm the conclusion obtained in Sec.II, i.e., the nonlinear interaction $b|\phi|^2\phi$ in Eq.(8) is caused by interaction among the particles or between the particle and background field, when the real motions of the particles and background as well as their interactions are completely considered.

In order to use fourth–order Runge-Kutta method[51,60] to solve numerically equation (108) we must discretize Eqs.(108) and (109) by means of, which can be denoted as

$$i\hbar\dot{\phi}_n(t) = \varepsilon\phi_n(t) - J[\phi_{n+1}(t) + \phi_{n-1}(t)] + (\chi/2r_0)[u_{n+1}(t) - u_{n-1}(t)]\phi_n(t) \tag{111}$$

$$M\ddot{u}_n(t) = W[u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)] + (\chi/2r_o)[|\phi_{n+1}|^2 - |\phi_{n-1}|^2] \tag{112}$$

where the following transformation relation between continuous and discrete functions are used

$\phi(x,t) \to \phi_n(t)$ and $u(x,t) \to u_n(t)$

$$\phi_{n\pm1}(t) = \phi_n(t) \pm r_0\frac{\partial\phi_n(t)}{\partial x} + \frac{1}{2!}r_0^2\frac{\partial^2\phi_n(t)}{\partial x^2} \pm \ldots.$$

$$u_{n\pm1}(t) = u_n(t) \pm r_0\frac{\partial u_n(t)}{\partial x} + \frac{1}{2!}r_0^2\frac{\partial^2 u_n(t)}{\partial x^2} \pm \ldots. \tag{113}$$

here $\varepsilon = \hbar^2/mr_0^2, J = \hbar^2/2mr_0^2, W = Mv_0^2/r_0^2, r_0$ is distance between two neighbouring lattice points. If using transformation: $\phi_n \to \phi_n\exp(i\varepsilon t/\hbar)$ we can eliminate the term $\varepsilon\phi_n(t)$ in Eq. (111). Again making a transformation: $\phi_n(t) \to a_n(t) = a(t)r_n + ia(t)i_n$, then Eqs. (111)-(112) become

$$\hbar\dot{a}r_n = -J(ai_{n+1} + ai_{n-1}) + (\chi/2r_0)(u_{n+1} - u_{n-1})ai_n \tag{114}$$

$$-\hbar\,\dot{a}i_n = -J(ar_{n+1} + ar_{n-1}) + (\chi/2r_0)(u_{n+1} - u_{n-1})ar_n \tag{115}$$

$$\dot{u}_n = y_n/M \tag{116}$$

$$\dot{y}_n = W(u_{n+1} - 2u_n + u_{n-1}) + (\chi/2r_o)(ar_{n+1}^2 + ai_{n+1}^2 - ar_{n-1}^2 - ai_{n-1}^2) \tag{117}$$

$$|a_n|^2 = |ar_n|^2 + |ai_n|^2 = |\phi_n|^2 \tag{118}$$

where ar_n and ai_n are real and imaginary parts of a_n. Equations.(114)-(117) can determine states and behaviors of the microscopic particle. There are four equations for one structure unit. Therefore, for the quantum systems constructed by N structure units there are 4N associated equations. When the fourth-order Runge-Kutta method[60] is used to numerically calculate the solutions of the above equations we should discretize them. Thus let the time be denoted by j, the step length of the space variable is denoted by h in the above equations. An initial excitation is required in this calculation, and it is chosen as, $a_n(o)=A\text{sech}[(n-n_0)(\chi/2r_0)^2/4JW]$ (where A is the normalization constant) at the size n, for the applied lattice, $u_n(0)=y_n(0)=0$. In the numerical simulation it is required that the total energy and the norm (or particle number) of the system must be conserved. The one dimensional system is fixed, N is chosen to be N=200, and a time step size of 0.0195 is used in the simulations. Total numerical simulation is performed through data parallel algorithms and MALAB language. If we choose appropriately the values of the parameters in Eqs.(111)-(112) or (114)-(117) we can calculate the numerical solution of the associated equation by using the fourth-order Runge-Kutta method[60] in uniform and periodic systems, thus the changes of $|\phi_n(t)|^2 = |a_n(t)|^2$, which is probability or number density of the microscopic particles occurring at the nth structure unit, with increasing time and position in time-place can be also obtained. This result is shown in Fig.4. This figure shows that the amplitude of the solution can retain constancy, i.e., the solution of Eqs.(111)-(112) or Eq.(8) at V(x)=constant is very stable while in motion for a longer time period. Therefore, equations.(111)-(112) have exactly soliton solutions for the systems.

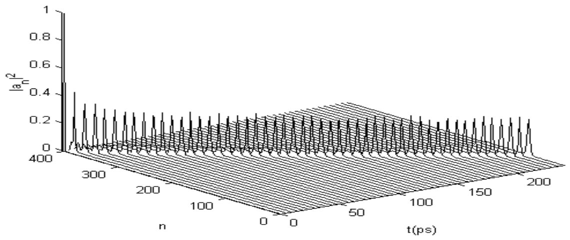

Fig4. *State of motion of the microscopic particle described by Eqs,(111)-(112)*

We further simulated numerically the collision behaviors of two particles described by nonlinear Schrödinger equations (8) at V(x) = $\varepsilon = \hbar^2 / mr_0$ =constant using the fourth-order Runge-Kutta method. This process resulting from two microscopic particles colliding head-on from opposite directions is shown in Fig.5. This figure manifests that the two particles can go through each other while retaining their form after the collision, which exhibits the corpuscle feature of the microscopic particles. The collision properties of microscopic particles described by the the nonlinear Schrödinger equation (8) are same with those obtained by Zakharov and Shabat[35-36], Desem and Chu[56-57], Tan et al[58] as well as Asossey et al[59]represented the hole-particle or dark spatial soliton of Eq. (11) at $b < 0$ as mentioned above. However, we see also from Fig.5 that a higher wave peak occurs in the colliding process, which is a result of superposition of two solitary waves and displays the wave feature of the microscopic particles.

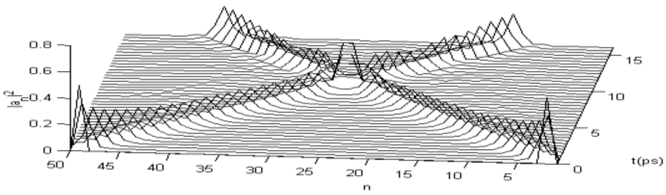

Fig5. *The features of collision of microscopic particles*

Thus we see clearly from figure 4 that the two particles can go through each other while retaining their form after the collision, which is the same with that of the classical particles, but the complicated interactions between the two particles in the colliding process occur, a combinatory wave with greater amplitude which is superposition of two wave, emerges. This manifests the wave feature of the particles. Therefore, the collision process of microscopic particles described by the the nonlinear Schrödinger equation (8) shows both an obvious corpuscle feature and a wave feature, i.e., wave-corpuscle duality, which are same with those in Sec.III.1 and Fig.2.

3.7. The Wave Behaviors of Microscopic Particles and Corresponding Uncertainty Relationship

3.7.1. The Wave Behaviors of the Microscopic Particles

As mentioned above, microscopic particles in the nonlinear quantum mechanics have also the wave property, in addition to the corpuscle property. This wave feature can be conjectured from the following reasons.

(1) Equation.(8) is wave equations and their solutions, Eqs.(19)-(21), (40)- (41) and as well as Eqs.(44),(46) and (47) are solitary waves having the features of traveling waves. A solitary wave consists of a carrier wave and an envelope wave, has certain amplitude, width, velocity, frequency, wavevector, and so on, and satisfies the principles of superposition of waves, although the latter are different from the classical waves or the de Broglie waves in the quantum mechanics.

(2) The solitary waves have reflection, transmission, scattering, diffraction and tunneling effects, just as that of classical waves or the de Broglie waves in the linear quantum mechanics. The diffraction and tunneling effects of microscopic particles will be studied in the following section. In this section we consider the reflection and transmission of the microscopic particles at an interface.

The propagation of microscopic particles (solitons) described by nonlinear Schrödinger equations (8) is different from that in uniform media. The nonuniformity can be due to a physical confining structure or two nonlinear materials being juxtaposed. One could expect that a portion of microscopic particles that was incident upon such an interface from one side would be reflected and a portion would be transmitted to the other side due to its wave feature. Lonngren *et al.*[61-63] observed the reflection and transmission of microscopic particles (solitons) in plasma consisting of a positive ion and a negative ion interface, simulated numerically the phenomena at the interface of two nonlinear materials and illustrate further the rules of reflection and transmission of microscopic particles.

Lonngren et al[61-63] simulated numerically the behaviors of solution of the nonlinear Schrödinger equation (11). They found that the signal had the property of a soliton. These results are in agreement with numerical investigations of similar problems by Aceves et al[64]. A sequence of pictures were obtained by Lonngren et al [61-63] at uniform temporal increments of the spatial evolution of the signal], which indicates that the incident wave propagating toward the interface between the two nonlinear media splits into a reflected and transmitted soliton at the interface. We think that the microscopic particles described by the nonlinear Schrödinger equation (11) possess also the wave feature. Thus we can used the following results of reflection and transmission of the solitary wave in Eq.(11) at interface to exhibit those of microscopic particle depicted by same equation. From the numerical values used in producing the figure, Lonngren et al[63] got the relationships of amplitudes and energy of the incident, the reflected and the transmitted solitons as well as the relationship between the reflection coefficient and the transmission coefficient. These rules are different from those for the KdV solitary wave and linear wave. Therefore, we can concluded that the wave feature of microscopic particles described by nonlinear Schrödinger equations (8) are different from those of both the KdV solitary wave and linear wave.

3.7.2. Correct form of Uncertainty Relation in the Linear Quantum Mechanics

As it is known, the microscopic particle has not a determinant position, disperses always in total space in a wave form in the linear quantum mechanics. Hence, the position and momentum of the microscopic particles cannot be simultaneously determined. This is just the well-known uncertainty relation. The uncertainty relation is an important formulae and also an important problem in the linear quantum mechanics that troubled many scientists. Whether this is an intrinsic property of microscopic particle or an artifact of the linear quantum mechanics or measuring instruments has been a long-lasting controversy. Obviously, it is closely related to elementary features of microscopic particles. Since we have established the nonlinear quantum mechanics, in which the natures of the microscopic particles occur considerable variations relative to that in the linear quantum mechanics, thus we expect that the uncertainty relation in nonlinear quantum mechanics could be changed relative to that in the linear quantum mechanics. Then the significance and essence of the uncertainty relation can be revealed by comparing the results of linear and nonlinear quantum theories.

The uncertainty relation in the linear quantum mechanics can be obtained from[25-27,65]

$$I(\zeta')=\int\left|\left(\zeta'\Delta\hat{A}+i\Delta\hat{B}\right)\psi(\vec{r},t)\right|^2 d\vec{r}\geq 0$$

or

$$\bar{\hat{F}}(\zeta')=\int\psi^*(\vec{r},t)\hat{F}\left[\hat{A}(\vec{r},t),\hat{B}(\vec{r},t)\right]\psi(\vec{r},t)d\vec{r} \tag{119}$$

In the coordinate representation, A and B are operators of two physical quantities, for example, position and momentum, or energy and time, and satisfy the commutation relation $\left[\hat{A},\hat{B}\right]=i\hat{C}$, $\psi\left(\vec{r},t\right)$ and $\psi^*\left(\vec{r},t\right)$ are wave functions of the microscopic particle satisfying the Schrödinger equation (1) and its conjugate equation, respectively, $\hat{F}=\left(\Delta A\zeta'+\Delta B\right)^2$, ($\Delta\hat{A}=\hat{A}-\bar{A}$, $\Delta\hat{B}=\hat{B}-\bar{B}$, $\bar{A}$ and $\bar{B}$ are the average values of the physical quantities in the state denoted by $\psi\left(\vec{r},t\right)$), is an operator of physical quantity related to $\bar{A}$ and $\bar{B}$, ξ is a real parameter. After some simplifications, we can get from Eq. (119)

$$I=\bar{F}=\overline{\Delta\hat{A}^2}\zeta'^2+2\overline{\Delta\hat{A}\Delta\hat{B}}\zeta'+\overline{\Delta\hat{B}^2}\geq 0$$

or

$$\overline{\Delta\hat{A}^2}\zeta'^2+\bar{\hat{C}}\zeta'+\overline{\Delta\hat{B}^2}\geq 0 \tag{120}$$

Using mathematical identities, this can be written as

$$\overline{\Delta\hat{A}^2}\,\overline{\Delta\hat{B}^2}\geq\frac{\bar{\hat{C}}^2}{4} \tag{121}$$

This is the uncertainty relation which is often used in the linear quantum mechanics. From the above derivation we see that the uncertainty relation was obtained based on the fundamental hypotheses of the linear quantum mechanics, including properties of operators of the mechanical quantities, the state of particle represented by the wave function, which satisfies the Schrödinger equation(1), the concept of average values of mechanical quantities and the commutation relations and eigenequation of operators. Therefore, we can conclude that the uncertainty relation in Eq. (121) is a necessary result of the quantum mechanics. Since the linear quantum mechanics only describes the wave nature of microscopic particles, the uncertainty relation is a result of the wave feature of microscopic particles, and it inherits the wave nature of microscopic particles. This is why its coordinate and momentum cannot be determined simultaneously. This is an essential interpretation for the uncertainty relation Eq.(121) in the linear quantum mechanics. It is not related to measurement, but closely related to the linear quantum mechanics. In other words, if the linear quantum mechanics could correctly describe the states of microscopic particles, then the uncertainty relation should also reflect the peculiarities of microscopic particles.

Equation (120) can be written in the following form:

$$\hat{F}=\overline{\Delta\hat{A}^2}\left(\zeta'+\frac{\overline{\Delta\hat{A}\Delta\hat{B}}}{\overline{\Delta\hat{A}^2}}\right)^2+\overline{\Delta\hat{B}^2}-\frac{\left(\overline{\Delta\hat{A}\Delta\hat{B}}\right)^2}{\overline{\Delta\hat{A}^2}}\geq 0 \tag{122}$$

or

$$\overline{\Delta\hat{A}^2}\left(\zeta'+\frac{\bar{\hat{C}}}{4\overline{\Delta\hat{A}^2}}\right)^2+\overline{\Delta\hat{B}^2}-\frac{\left(\bar{\hat{C}}\right)^2}{4\overline{\Delta\hat{A}^2}}\geq 0 \tag{123}$$

This shows that $\overline{\Delta\hat{A}^2}\neq 0$, if $\left(\overline{\Delta\hat{A}\Delta\hat{B}}\right)^2$ or $\bar{\hat{C}}^2/4$ is not zero, else, we cannot obtain Eq. (121) and $\overline{\Delta\hat{A}^2}\,\overline{\Delta\hat{B}^2}>\overline{\left(\Delta A\Delta B\right)^2}$ because when $\overline{\Delta\hat{A}^2}=0$, Eq. (123) does not hold. Therefore, $\overline{\left(\Delta\hat{A}^2\right)}\neq 0$ is a necessary condition for the uncertainty relation Eq. (121), $\overline{\Delta\hat{A}^2}$ can approach zero, but cannot be equal to zero. Therefore, in the linear quantum mechanics, the right uncertainty relation should take the form:

$$\overline{\Delta\hat{A}^2\Delta\hat{B}^2}>\left(\bar{\hat{C}}^2\right)^2/4 \tag{124}$$

3.7.3. Uncertainty Relation of Microscopic Particle Depicted by Nonlinear Schrödinger Equation

We now return to study the uncertainty relation of the microscopic particles described by nonlinear Schrödinger equation (8). In such a case the microscopic particles is a soliton and have a wave-corpuscle duality. Thus we have the reasons to believe that the uncertainty relation in this case should be different from Eq. (124) in the linear quantum theory .

We now derive this relation for position and momentum of a microscopic particle depicted by the nonlinear Schrödinger Equation (11) with a solution, ϕ_s, as given in Eq. (20). The function $\phi_s\left(x',t'\right)$ is a square integral function localized at $x_0'\neq 0$ in the coordinate space. The Fourier transform of this function is given by

$$\phi_s\left(p',t'\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi_s\left(x',t'\right)e^{-ip'x'}dx' \tag{125}$$

Using Eq. (20), then the Fourier transform is explicitly represented as

$$\phi_s\left(p,t'\right)=\sqrt{\frac{\pi}{2}}\,\mathrm{sech}\left[\frac{\pi}{4\sqrt{2}\eta}\left(p'-2\sqrt{2}\xi\right)\right]\exp\{i4(\eta^2+\xi^2-p'\xi/2\sqrt{2})t'-i\left(p'-2\sqrt{2}\xi\right)x_0'+i\theta\} \tag{126}$$

It shows that $\phi_s\left(p',t'\right)$ is also localized at p in momentum space. Equations (20) and (126) show that the microscopic particle is localized in the shape of soliton not only in position space but also in the momentum space. For convenience, we introduce the normalization coefficient B_0 in Eqs. (20) and (126), then obviously $B_o^2 = \eta / 4\sqrt{2}$, the position of the mass center of the microscopic particle, $\langle x' \rangle$, and its square, $\langle x'^2 \rangle, at \;\; t' = 0$ are given by

$$\langle x' \rangle = \int_{-\infty}^{\infty} x' \left|\phi_s(x')\right|^2 dx', \quad \langle x'^2 \rangle = \int_{-\infty}^{\infty} x'^2 \left|\phi_s(x')\right|^2 dx'. \tag{127}$$

We can thus find that

$$\langle x' \rangle = 4\sqrt{2}\eta A_0^2 x_0', \quad \langle x'^2 \rangle = \frac{A_0^2 \pi^2}{12\sqrt{2}\eta} + 4\sqrt{2} A_0^2 \eta x_0'^2 \tag{128}$$

respectively. Similarly, the momentum of the mass center of the microscopic particle, $\langle p' \rangle$, and its square, $\langle p'^2 \rangle$, are given by

$$\langle p' \rangle = \int_{-\infty}^{\infty} p' \left|\hat{\phi}_s(p')\right|^2 dp', \quad \langle p'^2 \rangle = \int_{-\infty}^{\infty} p'^2 \left|\hat{\phi}_s(p')\right|^2 dp' \tag{129}$$

which yield

$$\langle p' \rangle = 16 A_0^2 \eta \xi, \quad \langle p'^2 \rangle = \frac{32\sqrt{2}}{3} A_0^2 \eta^3 + 32\sqrt{2} A_0^2 \eta \xi^3 \tag{130}$$

The standard deviations of position $\Delta x' = \sqrt{\langle x'^2 \rangle - \langle x' \rangle^2}$ and momentum $\Delta p' = \sqrt{\langle p'^2 \rangle - \langle p' \rangle^2}$ are given by

$$\begin{aligned} (\Delta x')^2 &= A_0^2 \left[\frac{\pi^2}{12\eta} + 4\eta x_0'^2 \left(1 - 4\sqrt{2}\eta A_0^2\right)\right] = \frac{\pi^2}{96\eta^2}, \\ (\Delta p')^2 &= 32\sqrt{2} A_0^2 \left[\frac{1}{3}\eta^3 + \eta \xi^3 \left(1 - 4\sqrt{2}\eta A_0^2\right)\right] = \frac{8}{3}\eta^2, \end{aligned} \tag{131}$$

respectively. Thus we obtain the uncertainty relation between position and momentum for the microscopic particle depicted by the nonlinear Schrödinger equation in Eq. (11)

$$\Delta x' \Delta p' = \frac{\pi}{6} \tag{132}$$

This result is not related to the features of the microscopic particle (soliton) depicted by the nonlinear Schrödinger equation because Eq. (132) has nothing to do with characteristic parameters of the nonlinear Schrödinger equation. π in Eq. (132) comes from of the integral coefficient $1/\sqrt{2\pi}$. For a quantized microscopic particle, π in Eq. (132) should be replaced by $\pi\hbar$, because Eq. (125) is replaced by

$$\phi_s(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx \phi_s(x,t) e^{-ipx/\hbar}. \tag{133}$$

Thus the corresponding uncertainty relation of the quantum microscopic particle is given by

$$\Delta x \Delta p = \frac{\pi\hbar}{6} = \frac{h}{12} \tag{134}$$

This uncertainty principle also suggests that the position and momentum of the microscopic particle can be simultaneously determined in a certain degree. It is possible to estimate roughly the sizes of the uncertainty of these physical quantities. If it is required that $\phi_s(x',t')$ in Eq. (20) or $\phi_s(p',t')$ in Eq.(155) satisfies the admissibility condition *i.e.*, $\phi_s(0) \approx 0$, we choose $\xi = 140$, $\eta = \sqrt{300/0.253}/2\sqrt{2}$

and $\vec{x}_0^{'} = 0$ in Eq. (20) (In fact, in such a case we can get $\phi_s(0) \approx 10^{-7}$, thus the admissibility condition can be satisfied). We then get $\Delta x' \approx 0.02624$ and $\Delta p' \approx 19.893$, according to (133) and (134). This result shows that the position and momentum of the microscopic particles described by nonlinear Schrödinger equation could be determined simultaneously within a certain approximation, one of these cannot approach infinite.

Also, the uncertainty relation in Eq. (134) or Eq. (132) differ from the $\Delta x \Delta p > \hbar/2$ in Eq. (134) in the linear quantum mechanics. However, the minimum value $\Delta x \Delta p = \hbar/2$ has not been obtained from both the solutions of linear Schrödinger equation and experimental measurement up to now, except for the coherent and squeezed states of microscopic particles. Therefore we can draw a conclusion that the minimum uncertainty relationship is a nonlinear effect, instead of linear effect, and a result of wave-corpuscle duality.

From this result we see that when the microscopic particles satisfy $\Delta x \Delta p > \hbar/2$, then their motions obey laws of the linear quantum mechanics, the particles are some waves. When the uncertainty relationship of $\Delta x \Delta p = \hbar/12 \; or \; \pi/6$ is satisfied, the microscopic particles should be described by nonlinear Schrödinger equation (8), and have a wave-corpuscle duality. If the position and momentum of the particles meets $\Delta x \Delta p = 0$, then the particles have only a corpuscle feature, i.e., they are the classical particles. Therefore, the minimum uncertainty relation in Eq.(134) and (132) exhibits clearly the wave-corpuscle duality of microscopic particles described by nonlinear Schrödinger equation, which bridges also the gap between the classical and linear quantum mechanics. This is a very interesting result in physics.

3.7.4. The Uncertainty Relations of the Coherent States

As a matter of fact, we can represent one-quantum coherent state of harmonic oscillator by[66]

$$|\alpha\rangle = \exp\left(\alpha\hat{b}^{+} - \alpha^{*}\hat{b}\right)|0\rangle = e^{-\alpha^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n-1}}\hat{b}^{+n}|0\rangle,$$

in the number picture, which is a coherent superposition of a large number of microscopic particles (quanta). Thus

$$\langle\alpha|\hat{x}|\alpha\rangle = \sqrt{\frac{\hbar}{2\omega m}}\left(\alpha+\alpha^{*}\right), \quad \langle\alpha|\hat{p}|\alpha\rangle = i\sqrt{\hbar m\omega}\left(\alpha-\alpha^{*}\right),$$

and

$$\langle\alpha|\hat{x}^2|\alpha\rangle = \frac{\hbar}{2\omega m}\left(\alpha^{*2}+\alpha^2+2\alpha\alpha^{*}+1\right), \quad \langle\alpha|\hat{p}^2|\alpha\rangle = \frac{\hbar\omega m}{2}\left(\alpha^{*2}+\alpha^2-2\alpha\alpha^{*}-1\right),$$

where

$$\hat{x} = \sqrt{\frac{\hbar}{2\omega m}}\left(\hat{b}+\hat{b}^{+}\right), \quad \hat{p} = i\sqrt{\frac{\hbar\omega m}{2}}\left(\hat{b}^{+}-\hat{b}\right),$$

and $\hat{b}^{+}\left(\hat{b}\right)$ is the creation (annihilation) operator of microscopic particle (quantum), α and α^{*} are some unknown functions, ω is the frequency of the particle, m is its mass. Thus we can get

$$\left(\Delta x\right)^2 = \frac{\hbar}{2\omega m}, \quad \left(\Delta p\right)^2 = \frac{\hbar\omega m}{2}, \quad \langle\Delta x\rangle^2\langle\Delta p\rangle^2 = \frac{\hbar^2}{4} \tag{135}$$

$$\frac{\Delta x}{\Delta p} = \frac{1}{\omega m}, \text{ or } \Delta p = \left(\omega m\right)\Delta x$$

This is a minimum uncertainty relationship for the coherent state.

For the squeezed state of the microscopic particle: $|\beta\rangle = \exp\left[\beta\left(b^{+2}-b^2\right)\right]|0\rangle$, which is a two quanta coherent state, we can find that

$$\langle\beta|\Delta x^2|\beta\rangle=\frac{\hbar}{2m\omega}e^{4\beta},\langle\beta|\Delta p^2|\beta\rangle=\frac{\hbar m\omega}{2}e^{-4\beta},$$

using a similar approach as the above . Here β is the squeezed coefficient[60-63] and $|\beta|<1$. Thus,

$$\Delta x\Delta p=\frac{\hbar}{2},\ \frac{\Delta x}{\Delta p}=\frac{1}{m\omega}e^{8\beta},\ \text{or}\ \Delta p=\Delta x(\omega m)e^{-8\beta} \tag{136}$$

This shows that the squeezed state meets a minimum uncertainty relationship, the momentum of the microscopic particle (quantum) is squeezed in the two-quanta coherent state compared to that in the one-quantum coherent state.

On the other hand, the minimum uncertainty relationship of coherent state is not changed with variation of time. In fact, according to quantum theory, the coherent state of a harmonic oscillator at time t can be represented by

$$|\alpha,t\rangle=e^{-i\hat{H}t}|\alpha\rangle=e^{-i\hbar\omega(\hat{b}+\hat{b}/2)t}|\alpha\rangle=e^{-i\hbar\omega t/2-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n e^{-i\hbar n\omega t}}{\sqrt{n!}}|n\rangle$$

$$=e^{-i\hbar\omega t/2}\left|\alpha e^{-i\hbar\omega t}\right\rangle,\quad \left(|n\rangle=\left(b^+\right)^n|0\rangle\right)$$

This shows that the shape of a coherent state can be retained during its motion, which is the same as that of a microscopic particle (soliton) in the nonlinear quantum mechanics. The mean position of the particle in the time-dependent coherent state is

$$\begin{aligned}\langle\alpha,t|x|\alpha,t\rangle&=\langle\alpha|e^{iHt/\hbar}xe^{-iHt/\hbar}|\alpha\rangle=\left\langle\alpha\left|x-\frac{it}{h}[x,H]+\frac{(-it)^2}{2!\hbar^2}[[x,H],H]+\cdots\right|\alpha\right\rangle\\&=\left\langle\alpha\left|x+\frac{pt}{m}-\frac{1}{2!}t^2\omega^2x+\cdots\right|\alpha\right\rangle=\left\langle\alpha\left|x\cos\omega t+\frac{p}{m\omega}\sin\omega t\right|\alpha\right\rangle=\sqrt{\frac{2\hbar}{m\omega}}|\alpha|\cos(\omega t+\theta)\end{aligned} \tag{137}$$

where $\theta=\tan^{-1}\left(\frac{y}{x}\right)$, $x+iy=\alpha$, $[x,H]=\frac{i\hbar p}{m}$, $[p,H]=-i\hbar m\omega^2x.$

Comparing Eq. (137) with the solution of a classical harmonic oscillator

$$x=\sqrt{\frac{2E}{m\omega^2}}\cos(\omega t+\theta),\quad E=\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2$$

we find that they are similar with

$$E=\hbar\omega\alpha^2=\langle\alpha|H|\alpha\rangle-\langle 0|H|0\rangle,\quad H=\hbar\omega\left(b^+b+\frac{1}{2}\right)$$

Thus, we can say that the mass center of the coherent state-packet indeed obeys the classical law of motion, which is also the same as the law of motion of microscopic particles described by nonlinear Schrödinger equation discussed in Eqs. (94)-(98).

At the same time, we can also obtain

$$\langle\alpha,t|p|\alpha,t\rangle=-\sqrt{2m\hbar\omega}|\alpha|\sin(\omega t+\theta),\langle\alpha,t|x^2|\alpha,t\rangle=\frac{2\hbar}{\omega m}\left[|\alpha|^2\cos^2(\omega t+\theta)+\frac{1}{4}\right],$$

$$\langle\alpha,t|p^2|\alpha,t\rangle=2m\hbar\omega\left[|\alpha|^2\sin^2(\omega t+\theta)+\frac{1}{4}\right]$$

Thus a time-dependent uncertainty relationship can be obtained and represented by

$$[\Delta x(t)]^2=\frac{\hbar}{2\omega m},\quad [\Delta p(t)]^2=\frac{1}{2}m\omega\hbar,\Delta x(t)\Delta p(t)=\frac{h}{2} \tag{138}$$

This is the same with Eq. (136). It shows that the minimal uncertainty principle for the coherent state is retained at all times, *i.e.*, the uncertainty relation does not change with time t. This result is also the same with those of microscopic particles described by nonlinear Schrödinger equation, but we see not these rules in the linear quantum mechanics.

The above results show that both one-quantum and two-quanta coherent states satisfy the minimal uncertainty principle. This is the same with those of the microscopic particles depicted by nonlinear Schrödinger equation. This means that coherent and squeezed states are a nonlinear quantum state, the coherence and squeezing of quanta are a kind of nonlinear quantum effect. Just so, the states of a microscopic particles described by the nonlinear Schrödinger equation (8), such as the Davydov's wave functions[67], both $\mathrm{ID}_1>$ and $\mathrm{ID}_2>$, and Pang's wave function of exciton-solitons[68-71] in protein molecules and acetanilide; the wave function of proton transfer in hydrogen-bonded systems and the BCS's wave function in superconductors[31], etc., are always represented by a coherent state. Hence, the coherence of particles does not belong to the systems described by linear quantum mechanics, because the coherent state cannot be obtained by superposition of linear waves, such as plane wave, de Broglie wave, or Bloch wave. Then the minimal uncertainty relation Eq. (134), as well as Eqs. (133) and (138), are only applicable to microscopic particles described by nonlinear Schrödinger equation. Thus it reflects the wave-corpuscle duality of the microscopic particles.

Also, the above results indicate not only the essences of nonlinear quantum effects of the coherent state or squeezing state but also that the minimal uncertainty relationship is an intrinsic feature of the nonlinear quantum mechanics systems depicted by nonlinear Schrödinger equation.

Pang *et al.*[68-71] also calculated the uncertainty relationship and quantum fluctuations and studied their properties in nonlinear electron-phonon systems based on the Holstein model by a new ansatz including the correlations among one-phonon coherent and two-phonon squeezing states and polaron state proposed by Pang. Many interesting results were obtained, such as the minimum uncertainty relationship is related to the properties of the microscopic particles. The results enhanced the understanding of the significance and essences of the minimum uncertainty relationship.

4. CONCLUSIONS, SHORTCOMINGS OF LINEAR QUANTUM MECHANICS AND ITS RESOLUTION OF DIFFICULTIES

4.1. The Essences and Limitations of the Linear Quantum Mechanics

As it is known, the quantum mechanics was the foundation of modern science, it triggered great successes in applications, especially on hydrogen atom and molecule as well as helium atom and molecule. However, the microscopic particles possess a wave feature and no corpuscle nature in quantum mechanics. Thus the existence at a point of the microscopic particles in time-space and their states are only represented by a probability, the sizes of mechanical quantities are expressed by some average values, conjugate physical quantities, such as coordinate and momentum, cannot determined simultaneously and meet an uncertainty relation, and so on. These features of microscopic particles concert not the traditional concept of particles and are contradictory with the experimental results of electronic diffraction on double seam by Davisson and Germer in 1927 and de Broglie's relation of wave-corpuscle duality[11-13]. Therefore, the probability description of the microscopic particles brings also about plenty of difficulties and troubles to understand the natures and essences of the linear quantum mechanics and results in an intense controversies in physics, which elongate and continue a century. Very surprisingly, these difficulties and controversies have not been solved up to now as described in Introduction. This displays sufficiently the limitations of the linear quantum mechanics. These limitations and shortages are embodied concretely in the basic hypotheses of linear quantum mechanics. The linearity of the theory, containing the linear superposition principle, linearity of dynamic equation and the independence of Hamiltonian operator of the systems on states of microscopic particles, and so on, are its great fault and defect. These features of quantum mechanics are inconsistent with practical cases. In the applications it cannot be used to give the true properties and rules of the microscopic particles in the systems of many particles and many bodies. When the quantum mechanics is used to study the properties of motion of microscopic particles in these complicated systems, we have to use an average potential to replace the complicated and real nonlinear interaction among these particles, or between the particle and backgrounds through using some approximate methods, such as, the signal and free electronic approximations, compact-binding approximation and average field approximation, and so on. In such a case we obtained only some approximate results from linear quantum mechanics[1-13,25-27]. Therefore the Schrödinger equation (1) is a linear dynamic equation, quantum mechanics is also only a linear and approximate theory, and can be used to describe only the motions of microscopic particles in simple systems, such as hydrogen atom and molecules as well as helium atom, in which the nonlinear interaction is very small.

Another great defects of linear quantum mechanics is to neglect the truthful motions of each microscopic particles and background field as well as true interactions between them. Thus plenty of complicated and nonlinear interactions among the particles or between the particle and background field have been completely blotted out, we used only an average interaction unrelated to the states of particles to investigate the properties of microscopic particles. This results in the independence of the Hamiltonian operator of the system on the states of particles, then the states of particles are determined by the kinetic energy term in Eq.(1). This determines that the microscopic particles have only a wave feature in quantum mechanics. Therefore, quantum mechanics is only an approximate theory and cannot obtain in truth the properties and states of motion of microscopic particles in complicated systems. This is just the essence and limitation of linear quantum mechanics.

4.2. Successes of the New Theory and Resolution of Difficulties of Quantum Mechanics

On the contrary, we have broken through the hypothesis of independence of Hamiltonian operator of the systems on states of microscopic particles, forsaken the above linearity hypothesis of linear quantum mechanics and taken into account the true motions of each particle and background field and the interactions between them, thus the microscopic particles accepted a nonlinear interaction and their laws of motion are then described by Eq.(8). This means that the significance of the wave function generates essential variations, thus natures and properties of the microscopic particles appear also considerable changes, when compared with those in linear quantum mechanics. The changes can be summarized as follows[17-37].

(1) In this new theory although the states of microscopic particles are still represented as a wave function $\phi(\vec{r},t)$ in Eq. (12), its absolute square, $|\phi(\vec{r},t)|^2 = |\varphi(\vec{r},t)|^2 = \rho(\vec{r},t)$, denotes no longer the probability of finding the microscopic particle at a given point in the space-time, but represents the mass density of the microscopic particles at that point. From this representation we can find out the particle number or the mass of the particle by $\int_{-\infty}^{\infty} |\phi|^2 d\tau = N$. This means that the concept of probability is abandoned thoroughly in the new theory. Then the difficulty for probability interpretation for the wave function of microscopic particle in quantum mechanics is solved.

(2) The dynamic equations the particles satisfy are not the Schrödinger equation (1), but nonlinear Schrödinger equation (8). Their solutions have a wave-corpuscle duality, which is embedded by organic combination of envelope and carrier wave as shown in Fig.2. In such a case the particle has not only wave features, such as a certain amplitude, velocity, frequency, and wavevector, but also corpuscle natures, such as, a determinant mass centre, size, mass, momentum and energy. This is the first time to explain the wave-corpuscle duality of microscopic particles in quantum systems. This is a great advance of modern quantum theory, thus the new theory can solve the difficulties in quantum mechanics, which existed about one century.

At the same time, we proved that the wave-corpuscle duality of microscopic particles is quite stable, even though an externally applied potential fields exist. Meanwhile we obtained that the corpuscle feature of microscopic particles differs from the classical particles, the wave feature differs also from those of both linear wave, such as de Broglie wave in quantum mechanics, and KdV solitary wave. It is a special solitary wave, which can embody both wave feature and corpuscle feature of microscopic particles. Therefore, the new theory has nontrivial significances and is just the goal sought by physicists.

(3)In the new theory, $\int_{-\infty}^{\infty} \phi^* x \phi d\tau$, $\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \phi^* x \phi d\tau$, $\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} \phi^* x \phi d\tau$ and $\int \phi^* H \phi dx$ or $\langle \phi | H | \phi \rangle$ are no longer some average values of the physical quantities in quantum mechanics, but represent the position, velocity and acceleration of the mass center and energy of the microscopic particles, respectively, i.e., they have determinant values. Thus, the presentations of physical quantities in the new theory appear considerably the variations relative to those in quantum mechanics. Thus these results can solve the difficulties of representations of the physical quantities in quantum mechanics.

(4) The microscopic particles have determinant mass, momentum and energy, and obey universal conservation laws of mass, momentum, energy and angular momentum. Their motions satisfy not only nonlinear Schrödinger equations but also Newtonian law, Lagrangian equation and Hamilton equation. This amount to that the new theory bridges over the gap between the classical mechanics and quantum mechanics.

(5)The microscopic particles meet also the classical collision rule, when they collide with each other. Although these particles are deformed in the collision process which denotes its wave feature, they can still retain their form and amplitude to move towards after collision, where a phase shift occurs only. This feature denotes that the microscopic particles in the new theory possess both corpuscle and wave property, but the corpuscle property differs from classical particles.

(6) The position and momentum of the mass centre of microscopic particles are determinant, but their uncertainties obey only to a minimal uncertainty relation due to the wave-corpuscle duality, which is different from those in quantum mechanics. This means that the coordinate and momentum of microscopic particles may be simultaneously determined in the new theory at a certain degree. This amount also to that the new theory bridges over the gap between the classical mechanics and quantum mechanics.

These new and interesting natures and properties of the microscopic particles in the new theory are completely different from those of quantum mechanics. This make us see clearly the essences and limitations of quantum mechanics, which is only a special case of the new theory. Thus, it is very necessary to establish nonlinear quantum mechanics, which is a development of quantum mechanics and can solve the difficulties and problems existed in quantum mechanics about one century[7-9], and can promote the development of physics and enhance and raise the knowledge and recognition levels to the essences of microscopic matter. We can predict that nonlinear quantum mechanics has extensive applications in physics including condensed matter physics, chemistry, biology, polymers, and so on.

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