

Analytical Approach for the Solution of Thermodynamic Identities with Relativistic General Equation of State in a Mixture of Gases

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Abstract: *D.P. Mason and A.M.Kgathi integrated some thermodynamic identities for an ideal gas equation of state $p = nkT$ where p is the pressure, n is the particle number density, k is the Boltzmann constant and T is the absolute temperature. The present authors extend that work for a mixture of ideal gases with the general equation of state $p = p(n_i, T)$ as a special case and found the total energy density function (μ) and the entropy per unit volume (S) as a function of other thermodynamic variables n and T .*

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1. INTRODUCTION

In a collision – dominated equilibrium for a gas, two thermodynamic variables describe the thermodynamic state of the system completely. D.P.Mason and A.M.Kgathi[1] considered the thermodynamic variables of a relativistic gas in collision – dominated equilibrium as functions of the particle number density (n) and the entropy per unit volume (S). For a one component relativistic gas, the general form for a collision-dominated equilibrium distribution function is

$$f(x, p) = \left[\exp(-\alpha(x) - \beta_a(x) p^a) - \varepsilon \right]^{-1} \quad (1)$$

Where $\alpha = \frac{K}{kT}$, $\beta = \frac{1}{kT}$, k = Boltzmann's constant, K = Chemical potential,

T = absolute temperature, p^a is the four – momentum of a particle at a point x^a of space-time, $\beta^a = \beta u^a$, $u^a = u_k^a = u_D^a$, u_k^a and u_D^a are the kinematic and dynamic mean four-velocities of the gas, ε has different values for different distribution. For relativistic Bose – Einstein distribution $\varepsilon = +1$, Fermi–Dirac distribution $\varepsilon = -1$, Maxwell–Boltzmann distribution $\varepsilon = 0$. Actually Oliver and Davis[2] have shown that the absolute temperature T is a homogeneous function of degree one in n and S in a relativistic gas in collision – dominated equilibrium with an equation of state $p = \mu$ where p is the isotropic pressure and μ is the total energy density. But Mason and Kgathi took the interesting results of Oliver and Davis in a different way and extended their

results to the equation of state other than $p = \mu$ for a one – component gas. In a recent paper Mason and Kgathi [1] have shown that for all values of ε in the equilibrium distribution function (1) in a one component gas the following two well-known thermodynamic identities

$$d\mu = TdS + Kdn \tag{2}$$

And

$$\mu = TS + Kn - p \tag{3}$$

are satisfied and for relativistic Maxwell-Boltzmann distribution the above two identities can be integrated for a relativistic ideal gas equation of state

$$p = nkT \tag{4}$$

where μ is the total energy density, T is the temperature in Kelvin, n is the particle number density, K is the chemical potential per particle, p is the pressure, S is the entropy per unit volume and k is the Boltzmann constant. They obtained the general solution from equations (2) to (4) for the absolute temperature T(n,S), the chemical potential per particle K(n,S) and the total energy density function $\mu(n,S)$.

In this paper the present authors consider a mixture of gases with a more general equation of state

$$p = p(T, n_i) \tag{5}$$

While the thermodynamic identities take the form

$$d\mu = TdS + \sum_i K_i dn_i \tag{6}$$

And

$$\mu = TS + \sum_i K_i n_i - p \tag{7}$$

Where n_i and K_i are respectively the particle number density and chemical potential of the i -th gas while μ , T, S and p are as before. Finally the case of a mixture of ideal gases is considered as a special case. The principal aim of this paper is to solve a linear differential equation for obtaining the expression of the total energy density function (μ) and entropy per unit volume (S) by an analytical approach. This technique may be used in different branches of Physics and in engineering sciences.

In section 2, we write down the basic equations from the thermodynamic identities and found the expression for total energy density function (μ) and entropy per unit volume (S) after solution of equation (15). In section 3, mixture of ideal gases topic is discussed and expressions for μ and S for those mixture of ideal gases are given. The entire discussion is given in section 4. Section 5 contains conclusion and the future planning of the problem.

2. FORMATION OF THE DIFFERENTIAL EQUATION AND ITS SOLUTION

From the equation (6) one gets

$$\mu = \mu(S, n_1, n_2, \dots) \tag{8}$$

$$\frac{\partial \mu}{\partial S} = T \tag{9}$$

$$\frac{\partial \mu}{\partial n_i} = K_i \tag{10}$$

Hence in view of the equation (5), one can reduce the equation (7) to

$$\mu = S \frac{\partial \mu}{\partial S} + \sum_i n_i \frac{\partial \mu}{\partial n_i} - p \left(\frac{\partial \mu}{\partial S}, n_1, n_2, \dots \right) \quad (11)$$

p being a given function of the variables indicated the equation (11) is the equation to be solved for μ .

Now we define

$$m_1 = n_1, m_j = \frac{n_j}{n_1} \quad \text{for } j \geq 2 \quad (12)$$

$$\lambda(S, m_1, m_2, \dots) = \mu(S, n_1, n_2, \dots) \quad (13)$$

$$q(T, m_1, m_2, \dots) = p(T, n_1, n_2, \dots) \quad (14)$$

So that

$$\frac{\partial \mu}{\partial S} = \frac{\partial \lambda}{\partial S}$$

$$\frac{\partial \mu}{\partial n_1} = \frac{\partial \lambda}{\partial m_1} - \sum_{j=2,3,\dots} \frac{\partial \lambda}{\partial m_j} \frac{m_j}{m_1}$$

And

$$\frac{\partial \mu}{\partial n_j} = \frac{\partial \lambda}{\partial m_j} \frac{1}{m_1}$$

The equations (12) to (14) along with the last three relations reduce the equation (11) to

$$\lambda = S \frac{\partial \lambda}{\partial S} + m_1 \frac{\partial \lambda}{\partial m_1} - q \left(\frac{\partial \lambda}{\partial S}, m_1, m_2, \dots \right) \quad (15)$$

Where the equation (15) contains m_2, m_3 etc. as mere parameters. This equation can be solved in the following way:

Differentiating the equation (15) with respect to S while treating m_1, m_2, \dots as constants and using the equations (9) and (13) one gets

$$\left(S - \frac{\partial q}{\partial T} \right) \frac{\partial T}{\partial S} + m_1 \frac{\partial T}{\partial m_1} = 0$$

Now instead of treating S, m_1, m_2, \dots as independent if one treats T, m_1, m_2, \dots as independent variables the above equation takes the form

$$S - \frac{\partial q}{\partial T} = m_1 \frac{\partial S(T, m_1, m_2, \dots)}{\partial m_1}$$

This is a linear equation in S and is solved by integrating this equation. Thus the solution is

$$S = m_1 \frac{\partial P}{\partial T} \quad (16)$$

$$\text{Where } P = \phi(T, m_2, m_3, \dots) - \int \frac{q(T, m_1, m_2, \dots)}{m_1^2} dm_1 \quad (17)$$

Also integrating by parts the equation (9) and using the equations (16) and (17) one gets

$$\mu = m_1 \left(T \frac{\partial P}{\partial T} - P \right) + \alpha(m_1, m_2, \dots)$$

Where α is an arbitrary function of the variables indicated. Now putting the last equation back into the equation (11) and using the equations (9), (16) and (17) one gets

$$\alpha(m_1, m_2, \dots) = m_1 \cdot \text{constant}$$

Hence the solution for μ is given in the following form:

$$\mu = m_1 \left(T \frac{\partial P}{\partial T} - P \right) \tag{18}$$

But it is found from the equation (18) that α can be easily absorbed in the arbitrary function P where m_2, m_3, \dots are treated as constants, ϕ is an arbitrary function of the variables indicated, the function $q(T, m_1, m_2, \dots)$ is related to the given function $p(T, n_1, n_2, \dots)$ through the equation (14) while m_1, m_2, \dots are related to n_1, n_2, \dots through the equations (12). As in the previous equations (16) to (18) along with the equations (12) and (14), we express μ and S as functions of T, n_1, n_2, \dots and thereby we express μ as a function of S, n_1, n_2, \dots with T as the parameter.

3. MIXTURE OF IDEAL GASES [SPECIAL CASE]

Restricting to a mixture of ideal gases and using the law of partial pressure the equation (5) takes the form

$$p = kT \sum_{i=1,2} n_i \tag{19}$$

Where k is the Boltzmann constant. Putting this in equation (14) and using the equations (12), (13), (16), (17) and (18) one gets

$$S = n_1 \frac{\partial \phi \left(T, \frac{n_2}{n_1}, \frac{n_3}{n_1}, \dots \right)}{\partial T} - k \sum_i n_i \ln n_i \tag{20}$$

$$\mu = n_1 \left[T \frac{\partial \phi \left(T, \frac{n_2}{n_1}, \frac{n_3}{n_1}, \dots \right)}{\partial T} - \phi \left(T, \frac{n_2}{n_1}, \frac{n_3}{n_1}, \dots \right) \right] \tag{21}$$

Where ϕ is an arbitrary function of the variables indicated as before?

4. DISCUSSION

In a relativistic gas with collision – dominated equilibrium for equation of state $p = \mu$, Oliver and Davis took the prior work for the determination of the absolute temperature T and showed that T is a homogeneous function in n and S . Further it can be observed that the relativistic perfect gas law $p = nkT$ holds for the relativistic Maxwell-Boltzmann distribution function. But for a quantum gas, the equation of state (4) is not satisfied whereas for a non-quantum gas in collision-dominated equilibrium described by the relativistic Maxwell-Boltzmann distribution, the equation of state (4) is satisfied. Moreover Mason and Kgathi found the two independent thermodynamic variables (n and S) for the two thermodynamic identities (2) and (3) with the ideal gas equation of state (4). But this result is valid for a limited range of energies. After that the present authors integrated the same identities for a mixture of gases with a more general equation of state $p = p(n_i, T)$ for the variables μ, S and K . It is a more general result than that of Mason and Kgathi from which the result for a single gas is obtained as a special case.

5. CONCLUSION

In summary , the thermodynamic identities given by equations (6) and (7) for a mixture of gases whose equation of state is given by the equation (5) , have been integrated to give the equations (16) to (18) where $q (T , m_1 , m_2 , \dots)$ is connected to $p (T , n_1 , n_2 , \dots)$ through the equation (14) while m_1 , m_2 , \dots etc. are connected to n_1 , n_2 , \dots etc. through the equations (12). In the special case of a mixture of ideal gases where the equation of state is given by the equation (19), S and μ are given by the equations (20) and (21). This work is the generalized extension of the previous work of Mason and Kgathi[1] . Our future plan is to solve the same thermodynamic identities (2) and (3) for many component quantum gas with modified equation of state.

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REFERENCES

- [1] D.P.Mason and A.M.Kgathi , J.Math. Phys. 32 (1991) 493
- [2] D.R.Oliver and W.R.Davis, Ann. Inst. H. Poincare 30 (1979) 339