International Journal of Advanced Research in Physical Science (IJARPS)
Volume 1, Issue 1, May 2014, PP 10-15
www.arcjournals.org

New Perspective on Conservation Laws

Mukesh Kumar
Delhi,India
Physics.ssn@gmail.com

Abstract: *We focus on classical mechanical systems with a finite number of degrees of freedom and make no apriori assumption about the existence of Lagrangian, Hamiltonian or canonical momenta. Our work sheds new light on inverse problem of physics, Noether theorem inversion and symplectic canonical nature of classical phase space. The main results of this work are derived by the use of the Poisson Bracket, whose expression in local variables is given. Following our new approach, conserved quantities are related to Noether symmetries and Lie symmetries.*

1 INTRODUCTION

In this work we focus on classical mechanical systems with finite number of degrees of freedom. Conventionally a dynamical system is represented in terms of a Lagrangian function and constants of the motion may be associated with symmetries of the Lagrangian [1] by application of Noether's theorem. While all this is well known , what is not well studied is a new approach in which the existence of Lagrangian is not assumed apriori. Our work, which makes no apriori assumption about the existence of Lagrangian, Hamiltonian or canonical momentum , will bring out new interconnections between inverse problem in classical mechanics, symplectic canonical nature of classical phase space, and Noether theorem inversion.

The inverse problem of classical mechanics conventionally involves a study of the constraints known as Helmholtz conditions [2] that generalized forces must satisfy in order for them to be derivable from a nonsingular Lagrangian. The set of Helmholtz conditions (Section 2) is intimately connected with the symplectic canonical structure of phase space (Section 3) which in turn has key bearing on the following considerations (Section 5) relating conserved quantities to Noether symmetries.

In recent years, there has been revival of interest in Lie symmetries which differ entirely from Noether symmetries. A Noether symmetry in the most general formulation of the Noether theorem in Lagrangian mechanics [3] is defined by the fact that under it the Lagrangian is transformed into a total time derivative thereby leaving the action integral invariant. Lie symmetries which are the continuous symmetries of differential equations have nothing at all to do with invariance of the action. We discuss the general conditions for the invariance of our equations of motion and prove conservation of a certain quantity φ (Section 4). The latter is shown to vanish for Noether symmetries (Section 5). That is,

φ is a conservation law for Lie symmetries ,

φ vanishes for Noether symmetries .

2 INVERSE PROBLEM OF PHYSICS

The inverse problem of classical mechanics deals with the problem of determining a Lagrangian formulation that is equivalent to Newtonian formulation of a dynamical system. The Helmholtz conditions for the existence of a nonsingular Lagrangian for an equation of motion of the form

$\Lambda^i = \ddot{x}^i - f^i(x, \dot{x}, t) = 0$ (1)

Are

$w_{ij} = w_{ji}$,(2)

$$\frac{\partial w_{ij}}{\partial \dot{x}^k}=\frac{\partial w_{ik}}{\partial \dot{x}^j},(3)$$

$$\Gamma(w_{ij})=-\frac{1}{2}[w_{ik}\frac{\partial f^k}{\partial \dot{x}^j}+w_{jk}\frac{\partial f^h}{\partial \dot{x}^j}],(4)$$

$$\Gamma(t_{ij})\equiv\frac{1}{2}\Gamma(w_{ik}\frac{\partial f^k}{\partial \dot{x}^j}-w_{jk}\frac{\partial f^k}{\partial \dot{x}^i})=w_{ik}\frac{\partial F^k}{\partial x^j}-w_{jk}\frac{\partial F^k}{\partial x^i},(5)$$

where the elements of the nonsingular matrix $w_{ij}(x^k,\dot{x}^k,t)$ are the integrating factors in the equation

$$w_{ij}\Lambda^j=\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}^i})-\frac{\partial L}{\partial x^i}(6)$$

and the vector field

$$\Gamma=\frac{\partial}{\partial t}+\dot{x}^i\frac{\partial}{\partial x^i}+f^i\frac{\partial}{\partial \dot{x}^i}(7)$$

represents the total time derivative along the trajectory of eq.(1). Two identities which may be further derived from the system of eqs.(2)-(5) are

$$\frac{\partial t_{ij}}{\partial \dot{x}^k}=\frac{\partial w_{jk}}{\partial x^i}-\frac{\partial w_{ik}}{\partial x^j},(8)$$

$$\frac{\partial t_{ij}}{\partial \dot{x}^k}+\frac{\partial t_{jk}}{\partial \dot{x}^i}+\frac{\partial t_{ik}}{\partial \dot{x}^j}=0.(9)$$

If the Helmholtz conditions are satisfied, the Lagrangian must exist ; this means that an N-dimensional dynamical system can be described by the Lagrangian $L(x^1,x^2,...x^N,\dot{x}^1,\dot{x}^2,...\dot{x}^N,t)$ and that the Eular Lagrange equations may be put in the form $\Lambda^i=0$. If the integrating factors are taken to be elements of the unity matrix, the Lagrangian is restricted to have the form

$$\frac{1}{2}\dot{x}^i\dot{x}^i-V(x,\dot{x},t) \quad .$$

While this is well known, a new perspective on the inverse problem of classical mechanics is provided in the following section where we discuss the Poisson Bracket axioms. We show that, locally at least, a configuration space with commuting coordinates leads to a symplectic canonical structure if Helmholtz conditions are satisfied.

3 POISSON BRACKETS

For a complete axiomatic characterization of the Poisson Brackets , the manifolds in which we are interested are called canonical manifolds. These manifolds are generalizations of the phase space of analytical dynamics with coordinates and canonically conjugate momenta. A canonical manifold is a pair (V,ε) of a connected Hausdroff space with points $p,q,\ldots$, which has a countable basis of open sets, a family ε of continuous real-valued functions $f,g,\ldots$ over V and a Poisson Bracket with the following properties:

(a) Antisymmetry : $\{f,g\}=-\{g,f\}$

(b) Bilinearity : $\{\lambda f+\mu g,h\}=\lambda\{f,h\}+\mu\{g,h\}$

$\{f,\lambda g+\mu h\}=\lambda\{f,g\}+\mu\{f,h\}$

(c) Leibnitz Rule I : $\{f,gh\}=g\{f,h\}+h\{f,g\}$

(d) $\{f,c\} = 0$

(e) Jacobi identity : $J(f,g,h)=\{f\{g,h\}\}+\{g\{h,f\}\}+\{h\{f,g\}\}=0$

(f) The tensor field define by $\wedge(df,dg)=\{f,g\}$ is nondegenarate .

The dimension n of $\vee$ is even :$n=2f$. To every point $p\in\vee$ there exists a neighbourhood $N(p)$ and n elements $f^1,f^2,\ldots,f^n\in\varepsilon$ such that $x^i=f^i(q)$ is a homomorphic mapping of $N(p)$ on an open set in R^n. $x^1,x^2,\ldots,x^n$ are local coordinates of the point p. ε forms a ring with respect to addition and multiplication of functions. ε contains the constants {c} as a subring. It is worth remarking that according to (a)-(d),(f) a canonical manifold has a symplectic structure. This symplectic structure is further restricted by the condition $J(f,g,h)=0$; This restriction is the Jacobi identity.

Most importantly it can also be shown that the symplectic structure is also subject to the following restriction $\Gamma\{f,g\}=\{\Gamma(f),g\}+\{f,\Gamma(g)\}$.

This restriction is the Leibnitz rule II.

In local coordinates we obtain the expression

$$\{f,g\}(q)=\sum_{ij}\eta^{ij}\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial q^j}$$

With c^∞ functions $\eta^{ji}=-\eta^{ji}=\{q^i,q^j\}$. Here f and g are the c^∞ functions of variables q^1,q^2,q^3, etc.

We will now give a brief proof of the following :

Let the generalized forces obey the constraints in order that may be derivable from a nonsingular Lagrangian. Locally, there exist coordinates $x^1,x^2,\ldots,x^N,\dot{x}^1,\dot{x}^2,\ldots,\dot{x}^N$ to every point q such that

$$\{f,g\}(q)=\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial \dot{x}^j}g^{ij}-f\leftrightarrow g+\frac{\partial f}{\partial \dot{x}^i}\frac{\partial g}{\partial \dot{x}^j}s^{ij}$$

for $q\in N(p)$, a suitable neighbourhood of p, where the matrix g^{ij} is the inverse of the integrating factors matrix w_{ij} defined in eq.(6) and $s^{ij}=g^{ik}g^{kl}t_{kl}$, where t_{kl} is defined in eq.(5).

By using the above expression of Poisson Bracket, it is very easy to prove eqs.(a)-(d). To prove the Jacobi identity, we consider all possiblities for f, g, h. Here we discuss the two cases out of the four possibilities [6] :

Case (1): $f=x^i \quad g=x^j \quad h=x^k$

Then we have

$$J(f,g,h)=g^{im}\frac{\partial g^{jk}}{\partial x^m}-g^{jm}\frac{\partial g^{ik}}{\partial x^m} \quad .$$

This proves the Jacobi identity $J(x,x,x)=0$ if we use eq.(3).

Case (2): $f=x^i \quad g=\dot{x}^j \quad h=\dot{x}^k$

This gives

$$J(f,g,h)=g^{ib}\frac{\partial g^{kj}}{\partial x^b}-s^{ib}\frac{\partial g^{kj}}{\partial \dot{x}^b}+g^{kl}\frac{\partial s^{ij}}{\partial x^l}-g^{ja}\frac{\partial g^{ki}}{\partial x^a}+s^{ja}\frac{\partial g^{ki}}{\partial \dot{x}^a} \quad .$$

It can be easily shown that this equation is equivalent to eq.(8), which proves that Jocabi identity $J(x,x,x)=0$

The proof of $J(x,x,x)=0$ is trivial.

The last case is proved by use of eq.(9) through a tedious but straightforward calculation.

Since

$$
\begin{aligned}
\Gamma\{f,g\} &= \{\Gamma(f),g\} + \{f,\Gamma(g)\} \\
&+ \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial \dot{x}^j}\left(-s^{ij} - \frac{\partial f^j}{\partial \dot{x}^k}g^{ik} + \Gamma(g^{ij})\right) + \frac{\partial f}{\partial \dot{x}^i}\frac{\partial g}{\partial x^j}\left(-s^{ij} + \frac{\partial f^i}{\partial \dot{x}^k}g^{kj} - \Gamma(g^{ij})\right) \\
&+ \frac{\partial f}{\partial \dot{x}^i}\frac{\partial g}{\partial \dot{x}^j}\left(\frac{\partial f^i}{\partial x^k}g^{kj} - \frac{\partial f^i}{\partial \dot{x}^k}s^{kj} + \frac{\partial f^j}{\partial x^k}g^{ik} - \frac{\partial f^j}{\partial \dot{x}^k}s^{ik} + \Gamma(s^{ij})\right),
\end{aligned}
$$

The Leibnitz rule II is proved if we invoke the eqs.(2)-(5).

We have elucidated the connection between the symplectic canonical structure of phase space (see axions (a) - (f)) and the Helmholtz conditions (see eqs.(2)-(5), (8) and (9)).

4 LIE SYMMETRIES AND CONSTANTS OF MOTION

In this section we discuss the Lie symmetries for the equations of motion (1) by utilizing Poisson Bracket properties.

Suppose that a differentiable one-parameter group is generated by

$$G=\eta^i(x,t)\frac{\partial}{\partial x^i}+\xi^i(x,t)\frac{\partial}{\partial t}$$

which leaves the equations of motion (1) invariant. The general conditions for the invariance of eq.(1) under the resulting infinitesimal transformations are

$$\cup''\Lambda^i=0$$

where the second extended operator $\cup''$ is given by

$$\cup''=\xi\frac{\partial}{\partial t}+\eta_i\frac{\partial}{\partial x^i}+\eta'^i\frac{\partial}{\partial x^i}+\eta''^i\frac{\partial}{\partial \ddot{x}^i}$$

with

$$\eta'^i=\Gamma(\eta^i)-\dot{x}^i\frac{d\xi}{dt}$$

$$\eta''^i=\Gamma(\eta'^i)-\ddot{x}^i\frac{d\xi}{dt}$$

It may be recalled that for the total time derivative of a function $\varphi(q.\square q,t)$ we may write

$$\Gamma(\varphi)=\frac{\partial\varphi}{\partial t}+\dot{x}^i\frac{\partial\varphi}{\partial x^i}+f^i\frac{\partial\varphi}{\partial x^i}$$

The first integrals for the eq.(1) may be constructed from the analysis of symmetry transformation vectors obtained by the application of the condition $\cup''\Lambda^i=0$ to the eq.(1). This approach relies on knowledge of a previously found constant of the motion.

Recently Hojman has presented a new conservation theorem constructed in terms of a symmetry transformation vector of the eq.(1) only; no previous knowledge of a constant of motion is needed. It can be shown that any Lie symmetry of eq.(1) determines a constant given by

$$\varphi=\frac{E(D)}{D}+\frac{\partial\eta^i}{\partial x^i}+\frac{\partial\eta^i}{\partial x^i}(10)$$

where D is determinant of the nonsingular matrix of integrating factors in defined in eq.(6), and $E=\eta^i(x,\Box x,t)\frac{\partial}{\partial x^i}+\Box\eta^i(x,\Box xt)\frac{\partial}{\partial\Box x^i}$ is the symmetry transformation vector of eq.(1). This may also be established by applying a conservation theorem given by Hojman [4] and subsequently generalized by Gonzalez-Gascon (using geometric techniques)[5] to Lagrangian systems.

Note that in arriving at the conserved quantity given by eq.(10) , while it is necessary to postulate the Poisson Bracket axioms we do not assume any apriori existance of Lagrangian. We show in the next section that for actual invariance of the action, φ vanishes. That is, if the symmetry group does preserve the action, the quantity φ vanishes.

5 NOETHER THEOREM INVERSION

The inverse Noether theorem deals with the conditions under which the inversion of the Noether theorem is possible. Namely, the conditions under which to a given constant of the motion, it is possible to relate as an infinitesimal Noether symmetry under which the action integral is invariant. To any constant of motion C there corresponds an infinitesimal Noether transformation:

$$\delta x^i=\in g^{ij}\frac{\partial C}{\partial x^j}=\in\{x^i,C\}(11)$$

Which transforms the Lagrangian of the system into a total derivative. eq.(11) and the condition that $\Gamma(C)=0$ are the necessary and sufficient conditions on the constants of motion for their being related to a symmetry group that leaves the action integral invariant. These conditions were first demonstrated by trying to invert the Noether theorem in the Lagrangian formulation and by studying which are the transformation properties of the Lagrangian under the infinitesimal transformations generated by the constants of motion. It is shown in ref.[3] that eq.(11) determines a Noether symmetry ,that is , there exists a function $K(x,\Box x,t)$ such that under this transformation the Lagrangian is transformed into dK/dt.

The constant of motion defined by eq.(10) when restricted to Noether symmetries yields the expression

$$\varphi=E(\ln D)+\frac{\partial\{x^i,C\}}{\partial x^i}+\frac{\partial\{x^i,C\}}{\partial x^i}$$

Here we have used

$$\eta^i=\{x^i,C\}$$

which gives

[Sorry. Ignored \begin{aligned} ... \end{aligned}]

Further differentiation yields

$$\ddot{\eta}^i=\{f^i,C\}$$

This result shows that E is a symmetry transformation vector of the equations of motion (1) in this case. We can go one step further and find the numerical value of φ in this case.

To evaluate φ, we use the expression for the Poisson Bracket in local coordinates which gives

$$\varphi=-\frac{\partial C}{\partial x^m}(g^{lm}w_{ij}\frac{\partial g^{ij}}{\partial x^l}+s^{km}w_{ij}\frac{\partial g^{ij}}{\partial x^k}-\frac{\partial g^{im}}{\partial x^i}-\frac{\partial g^{im}}{\partial x^i})+\frac{\partial C}{\partial x^m}(g^{km}w^{ij}\frac{\partial g^{ij}}{\partial x^k}-\frac{\partial g^{im}}{\partial x^i})$$

The coefficient of the $\frac{\partial C}{\partial x^i}$ vanishes if make use of the Jacobi identity $J(x^j, x^m, x^l)=0$. Similarly the coefficient of $\frac{\partial C}{\partial x^i}$ can be shown to vanish if we make use of Jacobi identity $J(x^j, x^m, x^l)=0$. Hence φ vanishes for Noother symmetries.In brief , without assuming apriori the existence of Lagrangian we have shown that the conserved quantity φ vanishes identically in the case of Noether symmetries .

6 CONCLUSIONS

We have developed a new approach for the construction of constants of motion for the eq.(1). Neither a Lagrangian nor a Hamiltonian structure of the second order differential system is assumed to exist apriori for getting the conservation laws. What we assume are the various properties for the Poisson bracket defined between any two c^∞ functions of x,x, and t.While it is not necessary to assume that

the given equations of motion can be derived from a variational principle , there is no need to overemphasize that the basis for Poisson Bracket axiomatics relied upon so heavily in this work is in fact the Halmholtz conditions whose validity guarantees the existence of a Lagrangian.

REFERENCES

[1] Crampin, M., Pirani, F.A.E., Applicable differential geometry, London Math. Society Lecture Notes Series, no. 59, Cambridge University Press, 1986.

[2] Jones, G.L. ,Symmetry and conservation laws of differential equations, Il Nuovo Cimento 112(1997),1053-1059.

[3] Cramareanu, M., Conservations laws generated by pseudosymmetries with applications ,The Seminar of Mechanics Differential Dynamics Systems, West University of Timisoara , no. 62,1998, available at http://.

[4] Marsden, J.E., Ratiu, T.S., Introduction to Mechanics and Symmetry, Texts in Applied mathematics : 17, Springer-Verlag N.Y.,1994.

[5] Santilli,R.M., Foundations of Theoretical Mechanics I: The Inverse Problem in Newtanian Mechanics , Texts and Monographs in Physics, Springer-Verlag, N.Y., 1978.

[6] Soni S.K., Kumar M.,On the transformation from Newtonian to Lagrangian description with Poisson bracket, Europhysics Letters , 68 No. 4 (2004) , 501-507 .

Author's Biography

Mukesh Kumar, born in New Delhi (India) on 12 October 1976, an honours graduate and postgraduate from University of Delhi. He was awarded Ph.D. by University of Delhi in 2005. He was awarded JRF ans SRF fellowship by CSIR. In his more than 9 years of teaching career at Swami Shraddhanad College, University of Delhi, he has taught various courses to undergraduate students. He is actively involved in research in the fields of classical mechanics. He has published many papers and organized conferences.